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ELEMENTS
OF THE
INFINITESIMAL CALCULUS

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ELEMENTS

OF THE

INFINITESIMAL CALCULUS

BY

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THIRD EDITION

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PREFACE.

THE following pages are intended to serve as an introductory manual of Infinitesimal Calculus for beginners generally, but more especially for students of Engineering and other branches of Applied Science.

As a logical foundation of the Infinitesimal Calculus the doctrine of Limits must be accepted as essential, but an attempt has been made at an early stage to accustom the reader to those principles and operations which are used in the practical applications of the subject.

The order in which the subject is developed, though differing from that of many text-books, is believed to be well calculated to meet the difficulties and secure the interest of the student.

In the present edition many changes and some additions have been made which it is hoped will bring the book into accord with present-day treatment and needs.

To supplement the ordinary Mathematical Tables I have added short tables which are intended to facilitate curve tracing as well as the rapid calculation of integrals, etc.

My thanks are due to Professor Murray Macneill for suggestions and for assistance in proof-reading and in the verification of examples.

G. H. CHANDLER.

MONTREAL, December, 1906.

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ELEMENTS

OF THE

INFINITESIMAL CALCULUS.

CHAPTER I.

LIMITS. INFINITESIMALS.

1. Constant. Variable. When a quantity remains unchanged while another quantity changes, the former is called a constant, the latter a variable.

2. Limit of a variable. If the value of a variable v approaches nearer and nearer to that of a constant a in such a way that their difference becomes and remains less in absolute * value than any given positive number, however small, v is said to approach a as a limit, and a is called the limit of v.

If $x \doteq 2$,† the definition implies that the absolute value of $x-2$ becomes and remains less than any positive number we choose to assign; e.g., it becomes $<10^{-3}$; it further

* The absolute (or arithmetical or numerical) value is the value without regard to sign. Parallel vertical lines are used to indicate an absolute relation; thus $|-2|=2$ expresses that the absolute value of -2 is 2. So also $|x|<|a|$ or $x|<|a$ indicates that the absolute value of x is less than that of a.

† The symbol $\doteq$ signifies an approach to a limit. Thus $x \doteq 2$ may be read: x approaches 2 (as a limit). If the words "as a limit" are not expressed, they must be always understood.

diminishes, becoming less than any smaller given positive number (say 10^{-6}), and so on. While $x \doteq 2$, $x-2$ cannot be zero; the definition does not imply that x acquires the value 2. If it does become 2, it is no longer approaching 2 as a limit.

Ex. 1. The value of $(x^2-4)/(x-2)$ is equal to that of $x+2$ when any number except 2 is substituted for x. When $x=2$ the fraction takes the form 0/0, an expression which is undefined and meaningless, but when $x \doteq 2$ the limit of the value of the fraction is equal to that of $x+2$, i.e., 4, or in symbols *

$$\pounds_{x \doteq 2}\left(\frac{x^2-4}{x-2}\right) = 4.$$

Similarly, if $y=2x+x^2$, $\pounds_{x \doteq 0}(y/x)=2$.

2. θ being the radian measure † of an acute angle, θ lies between $\sin\theta$ and $\tan\theta$. Hence, dividing each of these into $\sin\theta$, $\sin\theta/\theta$ lies between 1 and $\cos\theta$. But $\cos\theta \doteq 1$ when $\theta \doteq 0$. Hence

$$\pounds_{\theta \doteq 0}\left(\frac{\sin\theta}{\theta}\right) = 1$$

3. The sum of the terms of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ has a limit as the number of terms increases without bound.

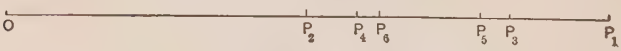

Fig. 1.

Let s_n stand for the sum of the first n terms. Take $OP_1=1$, $P_1P_2=-\frac{1}{2}$, $P_2P_3=\frac{1}{3}$, etc. Then $s_1=OP_1$, $s_2=OP_2$, $s_3=OP_3$, etc. The points with odd subscripts continue to move to the left,

* The symbol $\pounds$ is used for "the limit of" (Echols, *Differential and Int gral Calculus*).

† The radian ($=57°{\cdot}2958\ldots=206265''$) will be always understood to be the u. it angle unless the contrary is manifest.

those with even subscripts to the right; they tend toward meeting at some point P, since the fractions which are added $\doteq 0$. Hence $|OP-s_n|$ becomes and remains less than any given positive number, however small; therefore OP is the sum limit. The limit is that of the endless decimal ·69314 . . .

A series which has a limit is said to be convergent.

4. If the series $\frac{2}{1}-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\ldots$ be similarly represented, the two sets of points cannot come within a distance 1 of each other, since the absolute value of the fractions added $\doteq 1$. Suppose the even P's to approach a point P (really the point P of Ex. 3) as a limit of position. Then $|OP-s_n|$ may become less than any given positive number, however small, but it will not remain so, for the addition of a single term changes $|OP-s_n|$ by an amount approximately equal to 1. Consequently there is no limit, or the series is non-convergent. (Observe the significance of the words "and remains" in the definition of § 2).

3. A variable quantity may or may not be capable of assuming a value equal to that of the limit which it approaches. Thus in Ex. 1, $x+2\doteq 4$ when $x\doteq 2$, and $x+2=4$ when $x=2$. Also $(x^2-4)/(x-2)\doteq 4$ when $x\doteq 2$, but it cannot $=4$, for when $x=2$ there is no fraction. Again, the series $1-\frac{1}{2}+\frac{1}{3}-\ldots$ can never equal its limit. In certain cases the limit and the value are entirely different. For example (Ch. XL), $\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\ldots\doteq\frac{1}{2}\pi$ or $-\frac{1}{2}\pi$ according as $x\doteq\pi$ by increasing or decreasing, but $=0$ when $x=\pi$.

4. A variable may approach nearer and nearer to its limit

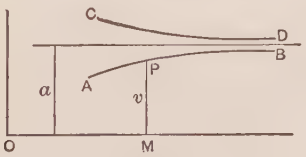

Fig. 2.

in three ways: (1) by increasing only, (2) by decreasing only, (3) and by increasing and decreasing. Thus, Fig. 2,

when M moves to the right v may be supposed to approach the limit a if P moves along AB only, (2) along CD only, or (3) from AB to CD, back to AB, etc., its value continually approaching a.

5. **Limit of a constant.** It is sometimes convenient to regard a constant as its own limit. Thus $\mathcal{L}_{x \doteq 0}(ax+b)=b$,* and hence if $a=0$, $\mathcal{L}_{x \doteq 0} b=b$. So also $\mathcal{L}_{x \doteq 0} ax/x=a$.

6. **Infinitesimals.** An infinitesimal is a variable whose limit is zero. It is therefore a quantity which approaches nearer and nearer to 0 in such a way that its absolute value becomes and remains less than any given positive number, however small. Thus x is infinitesimal when $x \doteq 0$; if x actually becomes 0 it is no longer infinitesimal.

Ex. When x is infinitesimal the following are also infinitesimals: x^2, $\sin x$, $\tan x$, $1-\cos x$, $\log(1+x)$, $1-2^x$.

7. Quantities which are infinitesimal are such solely on account of their tending to a limit zero, not because of their having arrived at any particular degree of smallness; in other words, their chief characteristic is not being small but getting smaller. Nevertheless, it is often convenient and sometimes necessary to suppose them very small when they begin to $\doteq 0$; hence it is customary to regard them as very small in all cases.

8. From the definition of § 2 it follows that the difference between a variable v and its limit a is an infinitesimal. Hence $v-a=i$, or $v=a+i$, where i is infinitesimal. Also if $v-a=i$, or $v=a+i$, where v is a variable, a a constant, and i an infinitesimal, a is the limit of v.

Notice that the sign of v is the same as that of a as soon as the absolute value of i becomes less than that of a.

9. **Infinites.** A variable which is increasing (or decreasing) without bound, i.e., so as to exceed in absolute value any

* As in algebra, letters near the beginning of the alphabet are used for constants unless the contrary is obvious.

given positive number, however large, is called an infinite, and is represented by ∞ (or $-\infty$). Such a quantity has no limit which accords with the definition of § 2.

It should be noticed that an infinite, like an infinitesimal, is a variable. If $x \doteq 0$, $1/x = \infty$ or $-\infty$, and if $x = \infty$* or $-\infty$, $1/x \doteq 0$.

Ex. When $x \doteq 2$, $x-2 \doteq 0$, and $1/(x-2) = \infty$ or $-\infty$ according as x approaches its limit by decreasing or increasing.

When $x \doteq \frac{1}{2}\pi$, $\tan x = \infty$ or $-\infty$ according as x is increasing or decreasing.

10. Quantities which are neither infinitesimal nor infinite are said to be finite. A finite variable is therefore one whose value stops short at some number which can be assigned.

11. If i is an infinitesimal and n any constant, ni is an infinitesimal.

For $|ni| <$ any assigned positive number a if $i < |a/n|$; but i does become $< |a/n|$, since a/n is also an assignable number. In special cases n may be 0, then ni remains $= 0$.

If n is infinite, ni may be infinite, or it may have a limit, which may or may not be zero.

12. The sum of any finite number n of infinitesimals is an infinitesimal.

Let i be a positive infinitesimal which is and remains greater than the absolute value of any of the given infinitesimals. Then the sum of the given infinitesimals $< ni$ and $> -ni$, and is therefore infinitesimal (§ 11).

If n is infinite, the sum may have a finite limit. The determination of such a limit is the fundamental problem of that part of the subject which is known as the Integral Calculus.

Ex. $\frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{n}{n^2} = \frac{1}{2}\frac{n(n+1)}{n^2} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$. The limit of the sum for n infinite is therefore $\frac{1}{2}$.

---

* $x = \infty$ should be read "x is a positive infinite," or "x increases without bound"; $x = -\infty$, "x is a negative infinite," or "x decreases without bound."

13. Propositions relating to limits. Let v_1, v_2 be two variables which have limits a_1, a_2. Then $v_1 = a_1 + i_1$ and $v_2 = a_2 + i_2$, where i_1 and i_2 are infinitesimals.

(A) The limit of the sum (or difference) of the variables is equal to the sum (or difference) of their limits.

For, $$(v_1 + v_2) - (a_1 + a_2) = i_1 + i_2,$$

which is infinitesimal. Hence $\mathcal{L}(v_1 + v_2) = a_1 + a_2$.

The proposition is evidently true for the sum of any finite number of variables.

(B) The limit of the product of the variables is equal to the product of their limits.

For, $$\begin{aligned} v_1 v_2 - a_1 a_2 &= (a_1 + i_1)(a_2 + i_2) - a_1 a_2 \\ &= a_2 i_1 + a_1 i_2 + i_1 i_2, \end{aligned}$$

which is infinitesimal. Hence $\mathcal{L} v_1 v_2 = a_1 a_2$.

This also is true for any finite number of variables. Thus if $x \doteq a$, $x^2 \doteq a^2$, $x^3 \doteq a^3$, etc.

(C) The limit of the quotient of the variables is equal to the quotient of their limits, provided that the limit of the divisor is not zero.

For, if $a_2 \neq 0$, $$\frac{v_1}{v_2} - \frac{a_1}{a_2} = \frac{a_1 + i_1}{a_2 + i_2} - \frac{a_1}{a_2} = \frac{a_2 i_1 - a_1 i_2}{a_2^2 + a_2 i_2},$$

which is infinitesimal, since the numerator $\doteq 0$, while the denominator $\doteq a_2^2$. Hence $\mathcal{L}\frac{v_1}{v_2} = \frac{a_1}{a_2}$.

If $a_2 = 0$ and $a_1 \neq 0$, v_1/v_2 is infinite and therefore has no limit. If $a_2 = 0$ and $a_1 = 0$, v_1/v_2 is the quotient of two infinitesimals and may have a limit, as in Exs. 1 and 2 of § 2. The determination of such a limit is the fundamental problem of that part of the subject which is known as the Differential Calculus.

Ex. 1. Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\therefore \dfrac{\tan\theta}{\theta} = \dfrac{\dfrac{\sin\theta}{\theta}}{\cos\theta}$.

Hence (§ 2, Ex. 2), $\mathcal{L}_{\theta \doteq 0}\left(\dfrac{\tan\theta}{\theta}\right) = \dfrac{1}{1} = 1$.

2. Since $1-\cos\theta = \dfrac{\sin^2\theta}{1+\cos\theta}$, $\therefore \dfrac{1-\cos\theta}{\theta^2} = \dfrac{\left(\dfrac{\sin\theta}{\theta}\right)^2}{1+\cos\theta}$;

$$\therefore \mathcal{L}_{\theta \doteq 0}\left(\frac{1-\cos\theta}{\theta^2}\right) = \frac{1}{1+1} = \frac{1}{2}.$$

3. $\tan\theta - \sin\theta = \dfrac{\sin^3\theta}{\cos\theta(1+\cos\theta)}$, $\therefore \mathcal{L}_{\theta \doteq 0}\left(\dfrac{\tan\theta-\sin\theta}{\theta^3}\right) = \dfrac{1}{2}$.

4. $\dfrac{n-1}{n-2} = \dfrac{1-\dfrac{1}{n}}{1-\dfrac{2}{n}}$, $\therefore \mathcal{L}_{n=\infty}\left(\dfrac{n-1}{n-2}\right)^n = 1$.

5. In any plane triangle, $\dfrac{a+b}{c} = \dfrac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}$.

If a and b are tangents at two points near * one another on a curve, and the points approach coincidence, A and $B \doteq 0$ and therefore the right-hand member of the above $\doteq 1$. Hence $\mathcal{L}(a+b)/c = 1$ when $c \doteq 0$. We may assume that the arc $> c$ and $< a+b$. Hence in any curve the limit of arc/chord is 1 as arc and chord $\doteq 0$.

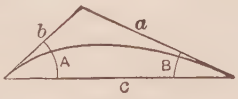

FIG. 3.

6. Show that $\mathcal{L}_{x \doteq a}\dfrac{3x^2-2x}{4x^3+x-2} = \dfrac{3a^2-2a}{4a^3+a-2}$, provided that a is not a root of the equation $4x^3+x-2=0$.

14. Orders of infinitesimals. If the limit of β/α, the quotient of two infinitesimals, is zero, β is said to be of a higher order than α. If β/α has a limit which is not zero, β is said to be of the same order as α. If β/α^n (where n is a constant) has a limit which is not zero, β is said to be of the nth order, α being assumed to be of the first order.

* It is assumed that the points may be taken so near each other that the arc is everywhere concave to the chord.

Ex. 1. α and β being of the first order, $\alpha\beta$ is of the second order, $\alpha^2\beta$, $\alpha\beta^2$ of the third, $\alpha^3\beta$, $\alpha^2\beta^2$, $\alpha\beta^3$ of the fourth.

2. If θ is infinitesimal and of the first order, $\sin\theta$ and $\tan\theta$ are of the first order, $1-\cos\theta$ of the second, $\tan\theta-\sin\theta$ of the third.

15. Definition. When the limit of the quotient of two infinitesimals is 1 the infinitesimals are said to be equivalent.*

Thus if θ is infinitesimal, θ, $\sin\theta$, and $\tan\theta$ are equivalent; also an infinitesimal arc and its chord.

If the limit of β/α is h (h being any constant, not zero), the limit of $\beta/h\alpha$ is 1, hence β is equivalent to $h\alpha$. Thus $1-\cos\theta$ is equivalent to $\frac{1}{2}\theta^2$, $\tan\theta-\sin\theta$ to $\frac{1}{2}\theta^3$.

16. If the difference of two infinitesimals of the same order is of a higher order, they are equivalent.

For, if $\beta=\alpha+i$, $\beta/\alpha=1+i/\alpha$, $\therefore\ \pounds(\beta/\alpha)=1$ if $\pounds(i/\alpha)=0$.

Conversely, the difference of two equivalent infinitesimals is of a higher order.

For, if $\beta/\alpha=1+i$, $\beta=\alpha+i\alpha$, and $i\alpha$ is of a higher order than α.

The letter I will be used as a symbol for higher infinitesimals. Thus if θ is infinitesmal, $\sin\theta=\theta+I$, $\tan\theta=\theta+I_1$; also, since $1-\cos\theta$ is of the second order, $\cos\theta=1+I_2$.

17. The limit of the quotient of two infinitesimals is not changed when either is replaced by an equivalent infinitesimal.

For
$$\frac{\beta}{\alpha}=\frac{\beta}{\beta'}\cdot\frac{\beta'}{\alpha'}\cdot\frac{\alpha'}{\alpha}.$$
$$\therefore\ \pounds\frac{\beta}{\alpha}=\pounds\frac{\beta'}{\alpha'}\quad\text{if}\quad\pounds\frac{\beta}{\beta'}=1\quad\text{and}\quad\pounds\frac{\alpha}{\alpha'}=1.$$

Hence if α and β consist of infinitesimals of different orders, the limit of β/α depends only on the infinitesimals of the lowest order in each.

---

* Not equal, but equivalent in the sense of being interchangeable in the determination of the limit of a quotient or of a sum (§§ 17, 91).

EXAMPLES.

1. Let AB be a circular arc of radius a subtending an infinitesimal angle θ at the centre, BC perpendicular to OA, AD and BE tangents. Let θ be regarded as of the first order. Then

(1) The arc $AB = a\theta$, and is therefore of the first order.

(2) $\mathop{\mathrm{Lt}}\dfrac{\text{chord } AB}{\theta} = \mathop{\mathrm{Lt}}\dfrac{2a \sin \frac{1}{2}\theta}{\theta} = \mathop{\mathrm{Lt}}\dfrac{2a \cdot \frac{1}{2}\theta}{\theta}$ (§§ 15, 17) $= a$.

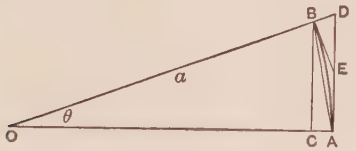

FIG. 4.

The chord is therefore of the first order and equivalent to $a\theta$, the arc.

(3) $\mathop{\mathrm{Lt}}\dfrac{CA}{\theta^2} = \mathop{\mathrm{Lt}}\dfrac{a(1-\cos\theta)}{\theta^2} = a \cdot \frac{1}{2}$.

Hence CA is of the second order and equivalent to $\frac{1}{2}a\theta^2$.

(4) $\mathop{\mathrm{Lt}}\dfrac{AD-CB}{\theta^3} = \mathop{\mathrm{Lt}}\dfrac{a(\tan\theta - \sin\theta)}{\theta^3} = a \cdot \frac{1}{2}$.

Hence $AD-CB$ is of the third order and equivalent to $\frac{1}{2}a\theta^3$.

(5) $\mathop{\mathrm{Lt}}\dfrac{BD-CA}{\theta^4} = \mathop{\mathrm{Lt}}\dfrac{(CA/\cos\theta)-CA}{\theta^4} = \mathop{\mathrm{Lt}}\dfrac{a(1-\cos\theta)^2}{\theta^4 \cos\theta}$

$= \mathop{\mathrm{Lt}}\dfrac{a(\frac{1}{2}\theta^2)^2}{\theta^4} = \frac{1}{4}a$.

Hence $BD-CA$ is of the fourth order and equivalent to $\frac{1}{4}a\theta^4$.

2. Show that the limit of $\dfrac{AD-CB}{ED-AE} = 2$.

3. Show that $(AE+EB)-$chord AB is equivalent to $\frac{1}{8}a\theta^3$, and hence that the difference of arc and chord is an infinitesimal of at least the third order.

4. Find $\mathop{\mathrm{Lt}}\dfrac{x-2}{x^3-x^2-2x}$ when $x \doteq 1$ and $x \doteq 2$. *Ans.* (1) $\frac{1}{2}$, (2) $\frac{1}{6}$.

Show that there is no limit when $x \doteq 0$ or -1.

5. $\pounds_{x \doteq 0} \dfrac{\operatorname{cosec} x - \cot x}{\sin x} = \frac{1}{2}$.

6. $\pounds_{\theta \doteq 0} \dfrac{1 - \cos^3 \theta}{\sin^2 \theta} = \frac{3}{2}$.

7. $\pounds_{n = \infty} \dfrac{(n-1)(n-3)}{n(n-2)} = 1$.

8. $\pounds_{x = -\infty} (2 + 3^x) = 2$.

9. $\pounds_{x \doteq 0} \dfrac{\sqrt{1+x} - 1}{x} = \frac{1}{2}$. [Rationalize the numerator.]

10. $\pounds_{x \doteq a} \dfrac{\sin x - \sin a}{x - a} = \cos a$.

CHAPTER II.

FUNCTIONS. DERIVATIVES. DIFFERENTIALS.

18. Function. When a variable quantity depends for its values upon those of another variable quantity, the first is said to be a *function* of the second, and the second is called the *variable* or *argument* of the first; e.g., x^2-2x+1, x^x, $\log(a+x)$, $\sin ax$, are functions of the variable x.

The expression $f(x)$ is used as a symbol for a function of x, $f(a)$ being the value of the function when $x=a$; e.g., if $f(x)=1-x^2$, $f(0)=1$, $f(1)=0$, $f(2)=-3$, $f(a)=1-a^2$.

For a similar purpose $F(x)$, $f'(x)$, etc., may be used, and $f(x, y)$, $F(x, y)$, etc., for functions of x and y.

The variable of a function may itself be a function of another variable, or it may be an *independent* variable—one to which arbitrary values may be assigned.

19. Implicit functions. In any equation containing two variables x and y, e.g., $y^2=4ax$, $\log(x+y)=2$, either of the variables is virtually or implicitly a function of the other, or an implicit function of the other, since the value of either is determined when that of the other is assigned. If we solve for y in terms of x, y becomes explicitly a function of x, or an explicit function of x.

20. Graphs. The curve whose equation is $y=f(x)$ is the graph or geometrical representation of the function $f(x)$. The ordinate corresponding to any abscissa x is the value of the function when the value of the variable is x.

When for a value of the variable there is only one corresponding value of the function, the function is said to be

single-valued. Thus e^x, Fig. 5, and $\log x$, Fig. 6, are single-valued functions. The function $\sin^{-1} x$, Fig. 7, is multiple-valued. It will in general be assumed that a function is single-valued; when such is not the case the function may be treated as single-valued by considering a limited range of its values. Thus $\sin^{-1} x$ is single-valued if values $> \frac{\pi}{2}$ and $< -\frac{\pi}{2}$ are excluded. The ordinate of the curve $y^2 = 4x$ is a double-valued function of x, but may be represented by two single-valued functions, $2\sqrt{x}$ and $-2\sqrt{x}$.

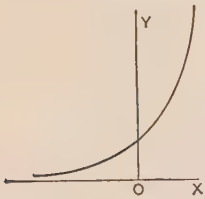

Fig. 5.

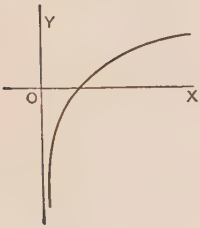

Fig. 6.

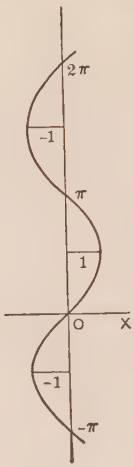

Fig. 7.

21. Continuity. In general a gradual change in the value of a variable produces a gradual change in the value of the function, but it is possible that a slight change in the variable may produce an abrupt finite or infinite change in the function. In more precise language, $f(x)$ is continuous for the value a of the variable when, as $h \doteq 0$ (h being a small change in x),

$$\pounds f(a+h) = \pounds f(a-h) = f(a),$$

and discontinuous if this relation is not true.

Let $OA = a$, $BA = AC = h$. In Fig. 8, $AP = f(a)$, $CR = f(a+h)$, $BQ = f(a-h)$. Also as $h \doteq 0$, $\pounds CR = AP$, and $\pounds BQ = AP$, hence the ordinate is continuous at A. But in Fig. 9, $\pounds CR = AP$, $\pounds BQ = AP'$; these are not equal, hence y is discontinuous at A. In Fig. 10, BQ becomes infinite when $h \doteq 0$, and y is discontinuous.

When the function changes abruptly from one finite value to another finite value it is said to have finite discontinuity.

when the function becomes infinite it is said to have infinite discontinuity.

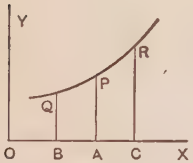

Fig. 8.

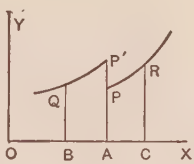

Fig. 9.

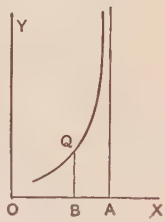

Fig. 10.

Ex. 1. $f(x) \equiv \dfrac{1}{2^{\frac{1}{x}}+1}$ has finite discontinuity at $x=0$. For, when $h \doteq 0$, $\pounds f(h)=0$ and $\pounds f(-h)=1$. Thus when x increases through the value 0 the function drops suddenly in value from slightly less than 1 to slightly more than 0, without passing through the intermediate values. It cannot be said to have any value when $x=0$.

2. The following have infinite discontinuity for $x=1$:

$$(x+1)/(x-1), \quad (x-1)^{-\frac{1}{2}}, \quad 3^{(1-x)^{-1}}, \quad \tan \tfrac{1}{2}\pi x.$$

3. Examine the function $2^{\frac{1}{x}}$ for $x=0$.

22. Interval. Values of the variable between two assigned values a and b are said to lie in the interval from a to b. The interval may be conveniently indicated by $[a, b]$. Reversal of a bracket indicates the exclusion of the adjacent end value; e.g., $[a, b[$ indicates values from a to b, including a but excluding b. Thus $(1-x^2)^{\frac{1}{2}}$ is real for the interval $[-1, 1]$, $(1-x^2)^{-\frac{1}{2}}$ is continuous for the interval $]-1, 1[$.

23. Increment. Any change in the value of a quantity is called an increment or difference of that quantity. An increment is positive or negative according as the quantity is increased or decreased. The symbols Δx, δx, are used for increments or differences of x.

24. Derivative. Let there be a variable and a function of that variable. A particular value of the variable being x

let the corresponding value of the function be y. Let x receive an increment Δx, and let Δy be the corresponding increment of y. The limit of $\Delta y/\Delta x$ when $\Delta x \doteq 0$ is called the derivative of y. Thus the derivative is the limit of the quotient of the increment of the value of the function by the increment of the value of the variable when the latter increment is infinitesimal. The primary object of the Infinitesimal Calculus is to determine this limit for various kinds of functions.

Ex. 1. If $y=x^2$, $\Delta y=(x+\Delta x)^2-x^2=2x\,\Delta x+(\Delta x)^2$.

$\therefore\ \Delta y/\Delta x=2x+\Delta x.\quad \therefore\ \mathcal{L}(\Delta y/\Delta x)=2x$ when $\mathcal{L}\Delta x=0$.

Similarly, if $y=ax^2$, $\mathcal{L}(\Delta y/\Delta x)=a\,.\,2x$, a being any constant.

2. If $y=ax^3$, show that $\mathcal{L}(\Delta y/\Delta x)=a\,.\,3x^2$.

3. If $y=x$, $\Delta y=\Delta x$, $\therefore\ \Delta y/\Delta x=1$, or $\mathcal{L}(\Delta y/\Delta x)=1$ (§ 5).

Similarly if $y=ax$, $\mathcal{L}(\Delta y/\Delta x)=a$.

4. $y=4x^3-3x^2+2x-1$.

The method of obtaining $\Delta y/\Delta x$ shows that the result will be the same as if each term were treated separately and the results added, also that a constant term disappears in subtracting.

$$\therefore\ \mathcal{L}(\Delta y/\Delta x)=4\,.\,3x^2-3\,.\,2x+2=12x^2-6x+2.$$

25. The general method illustrated in the above examples may be described as follows: Let $y=f(x)$. The value of the function for the value $x+\Delta x$ of the variable is $f(x+\Delta x)$; hence Δy, the increment of function, is $f(x+\Delta x)-f(x)$, and

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

This expression is simplified and its limit taken when $\Delta x \doteq 0$. This limit is the derivative, and is, for various values of x, a new function of x. It is called the derived function, or derivative function, or simply the derivative, of the given function. Let it be written $f'(x)$. Thus if $f(x)=x^2$, $f'(x)=2x$; if $f(x)=ax^3$, $f'(x)=3ax^2$; if $f(x)=x^2+2ax$, $f'(x)=2(x+a)$.

It should be noticed that there cannot be a limit (a derivative) unless $\Delta y \doteq 0$ as well as Δx; i.e., unless $\mathcal{L}f(x+\Delta x)=f(x)$ when $\Delta x \doteq 0$, or (§ 21) unless $f(x)$ is continuous for the value of x in question.

26. Geometrical illustration. Let $y=f(x)$ be the equation* of a curve of which $P(x, y)$ is a point. Then $\Delta y/\Delta x$ is the slope or gradient of the secant PQ. When $\Delta x \doteq 0$, Q approaches P, and the limit of position of the secant is (by definition) the tangent at P. Hence $\pounds(\Delta y/\Delta x)$ or $f'(x)$ is $\tan\phi$, the slope of the tangent at (x, y). Thus for the curve $y=x^2$, $\tan\phi=2x$; for $y=x^3$, $\tan\phi=3x^2$.

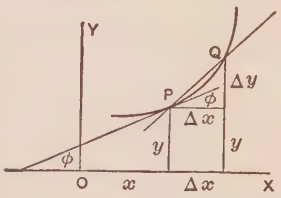

FIG. 11.

27. Differential. Def. The differential of the variable of a function is any increment of that variable; the differential of the function is the derivative of the function multiplied by the differential of the variable.

The letter d is used as an abbreviation for "the differential of." If then y or $f(x)$ is a function of x, the definition states that dx is any increment of x, and that dy or $df(x)$ is $f'(x)\,dx$.†

Hence $$\frac{dy}{dx}=f'(x)=\pounds\frac{\Delta y}{\Delta x} \text{ when } \Delta x \doteq 0.$$

Thus dx and dy are defined in such a way that dy/dx is equal to the limit of $\Delta y/\Delta x$, or the differential quotient is the limit of the difference or increment-quotient.

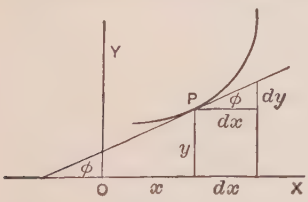

FIG. 12.

Geometrically, dy/dx is the slope ‡ of the tangent at (x, y), Fig. 12, and dy is the increment of the ordinate of the tangent corresponding to the increment dx of the abscissa. It should be noticed that, although dx may have any value, the value of dy/dx is independent of dx.

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* The angle between the axes is assumed to be a right angle in all cases unless the contrary is mentioned.

† From its position as a multiplier of dx the derivative $f'(x)$ is also called the differential coefficient of $f(x)$.

‡ If the angle between the axes is ω, $dy/dx=\sin\phi/\sin(\omega-\phi)$.

CHAPTER III.

DIFFERENTIAL OF A POWER, A PRODUCT, AND A QUOTIENT.

28. The operation of obtaining derivatives or differentials is called differentiation.

We now consider a few general formulæ which will assist in differentiating, first showing that the differential of the algebraical sum of any finite number of terms is equal to the algebraical sum of the differentials of the terms; also that a constant factor in a term appears as a factor in the differential of that term, and that a constant term disappears in differentiating, or has 0 for its differential.

Let $y=au+v-w+c$, where u, v, w are continuous functions of x, and a and c are constants. Let the increment Δx in x cause increments Δu, Δv, Δw in u, v, w, Δy being the resultant increment of y. Then

$$\begin{aligned}\Delta y &= [a(u+\Delta u)+(v+\Delta v)-(w+\Delta w)+c]-(au+v-w+c)\\ &= a\,\Delta u+\Delta v-\Delta w.\end{aligned}$$

$$\therefore \frac{\Delta y}{\Delta x}=a\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}-\frac{\Delta w}{\Delta x}.$$

Hence, taking the limits when $\Delta x=0$,

$$\frac{dy}{dx}=a\frac{du}{dx}+\frac{dv}{dx}-\frac{dw}{dx}, \quad \text{or} \quad dy=a\,du+dv-dw.$$

$$\therefore\ d(au+v-w+c)=a\,du+dv-dw.$$

29. Differential of a power. Let v be a function of x, to find $d(v^n)$, n being any constant. Let $y=v^n$. Then

$$\Delta y=(v+\Delta v)^n-v^n=v^n\left[\left(1+\frac{\Delta v}{v}\right)^n-1\right].$$

Taking $\Delta v < v$, expanding by the binomial theorem, and dividing by Δx,

$$\frac{\Delta y}{\Delta x} = nv^{n-1}\frac{\Delta v}{\Delta x}\left(1 + \frac{n-1}{1\,.\,2}\frac{\Delta v}{v} + \ldots\right).$$

Taking the limits when $\Delta x \doteq 0$, and $\therefore\ \Delta v$ also $\doteq 0$,

$$\frac{dy}{dx} = nv^{n-1}\frac{dv}{dx},$$

or
$$d(v^n) = nv^{n-1}\,dv. \qquad (A)$$

This result is true for all values of n. The case in which $n = \frac{1}{2}$ deserves special mention; (A) then becomes

$$d(\sqrt{v}) = \frac{dv}{2\sqrt{v}}. \qquad (B)$$

When $n = -1$, (A) becomes

$$d\left(\frac{1}{v}\right) = -\frac{dv}{v^2}. \qquad (B_1)$$

Ex. 1. $d(x^5) = 5x^4\,dx$.

2. $d(3x^5 + 2) = 3d(x^5) = 15x^4\,dx$.

3. $d(3x^4 - 2x^2 + 6) = 3\,.\,4x^3\,dx - 2\,.\,2x\,dx = 4x\,(3x^2 - 1)dx$.

4. $d\sqrt[3]{x^2} = d(x^{\frac{2}{3}}) = \frac{2}{3}x^{-\frac{1}{3}}\,dx$.

5. $d\left(\frac{1}{x^2}\right) = d(x^{-2}) = -2x^{-3}\,dx = -\frac{2\,dx}{x^3}$.

6. $d(a^2+x^2)^3 = 3(a^2+x^2)^2\,d(a^2+x^2)$
$= 3(a^2+x^2)^2\,2x\,dx = 6x(a^2+x^2)^2\,dx$.

In this example $v = a^2 + x^2$, and $n = 3$, a being constant.

7. $d\left(\frac{a-x}{a-b}\right) = \frac{1}{a-b}d(a-x) = -\frac{dx}{a-b}$.

8. $d\frac{1}{\sqrt{(ax+bx^2)^3}} = d(ax+bx^2)^{-\frac{3}{2}}$
$= -\frac{3}{2}(ax+bx^2)^{-\frac{3}{2}-1}\,d(ax+bx^2) = -\frac{3}{2}(ax+bx^2)^{-\frac{5}{2}}(a+2bx)\,dx$.

9. $d\sqrt{a^2-x^2} = \frac{d(a^2-x^2)}{2\sqrt{a^2-x^2}}$ by (B), $= \frac{-2x\,dx}{2\sqrt{a^2-x^2}} = -\frac{x\,dx}{\sqrt{a^2-x^2}}$.

10. $d\left(\frac{1}{a^2-x^2}\right) = -\frac{d(a^2-x^2)}{(a^2-x^2)^2}$ by (B_1), $= \frac{2x\,dx}{(a^2-x^2)^2}$.

30. Differential of a product. Let $y=uv$, where u and v are functions of x. Then

$$\Delta y=(u+\Delta u)(v+\Delta v)-uv=v\,\Delta u+u\,\Delta v+\Delta u\,\Delta v.$$

$$\therefore\ \frac{\Delta y}{\Delta x}=v\frac{\Delta u}{\Delta x}+u\frac{\Delta v}{\Delta x}+\Delta u\frac{\Delta v}{\Delta x}.$$

The limit of the last term is 0.

$$\therefore\ \frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx},$$

or
$$d(uv)=v\,du+u\,dv. \qquad (C)$$

Similarly,
$$d(uvw)=vw\,du+wu\,dv+uv\,dw. \qquad (C_1)$$

Ex. $d(4x+3)(x^2-1)=(x^2-1)d(4x+3)+(4x+3)d(x^2-1)$
$=(x^2-1)4\,dx+(4x+3)2x\,dx=2(6x^2+3x-2)dx.$

31. Differential of a quotient or fraction. Let the fraction be u/v, u and v both being variable. Then

$$d\left(\frac{u}{v}\right)=d\left(u\frac{1}{v}\right)=\frac{1}{v}\,du+u\left(-\frac{dv}{v^2}\right),\quad \text{by (C) and } (B_1).$$

Hence
$$d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}. \qquad (D)$$

Ex. $d\left(\frac{x^2-1}{x^2+1}\right)=\frac{(x^2+1)d(x^2-1)-(x^2-1)d(x^2+1)}{(x^2+1)^2}=\frac{4x\,dx}{(x^2+1)^2}.$

EXAMPLES.

1. $d(a^2-x^2)^3=-6x(a^2-x^2)^2dx.$
2. $d\sqrt{1+x^2}=x\,dx/\sqrt{1+x^2}.$
3. If $f(x)=ax^2+2bx+c$, $f'(x)=2(ax+b).$*
4. $y=\sqrt{x^3-a^3}$, $dy/dx=3x^2/2\sqrt{x^3-a^3}.$
5. $d[ax(x^2-1)(x+1)]=a(x+1)(4x^2-x-1)dx.$
6. $d\left(\frac{x^4-1}{x^4+1}\right)=\frac{8x^3\,dx}{(x^4+1)^2}.$
7. $d(1+x)\sqrt{1-x}=(1-3x)dx/2\sqrt{1-x}.$

---

* $f'(x)=df(x)/dx$, § 27.

8. $y=3(a+bx^2)^{\frac{5}{3}}$, $dy=10bx(a+bx^2)^{\frac{2}{3}}dx$.
9. $y=\sqrt{a^2+(b-x)^2}$, $dy=(x-b)dx/\sqrt{a^2+(b-x)^2}$.
10. $y=(a^3-x^3)^{-1}$, $dy/dx=3x^2/(a^3-x^3)^2$.
11. $y=x^3/(a+x)^2$, $dy/dx=(3a+x)x^2/(a+x)^3$.
12. $y=\sqrt{a^2-x^2}$, $dy=-x\,dx/\sqrt{a^2-x^2}$.
13. $y=\sqrt{2ax-x^2}$, $dy=(a-x)dx/\sqrt{2ax-x^2}$.
14. $y=x^3/(a^2-x^2)$, $dy/dx=(3a^2-x^2)x^2/(a^2-x^2)^2$.
15. $d[x^n/(1+x)^n]=nx^{n-1}\,dx/(1+x)^{n+1}$.
16. $d(a^2-x^2)^{-1}=2x\,dx/(a^2-x^2)^2$.
17. $y=x^2/\sqrt{1+x^4}$, $dy/dx=2x/(1+x^4)^{\frac{3}{2}}$.
18. $y=(x-a)/\sqrt{x}$, $dy/dx=(a+x)/2\sqrt{x^3}$.
19. $y=ax/\sqrt{2ax-x^2}$, $dy/dx=a^2x/(2ax-x^2)^{\frac{3}{2}}$.
20. $y=\sqrt{\dfrac{a+x}{a-x}}$, $dy=\dfrac{a\,dx}{(a-x)\sqrt{a^2-x^2}}$.
21. $y=2x/\sqrt{a^2+x^2}$, $dy=2a^2\,dx/(a^2+x^2)^{\frac{3}{2}}$.
22. $y=x(a^2+x^2)\sqrt{a^2-x^2}$, $dy/dx=(a^4+a^2x^2-4x^4)/\sqrt{a^2-x^2}$.
23. $f(x)=\sqrt{x+\sqrt{1+x^2}}$, $f'(x)=\sqrt{x+\sqrt{1+x^2}}/2\sqrt{1+x^2}$.
24. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Differentiating each term,

$$\frac{2x\,dx}{a^2}+\frac{2y\,dy}{b^2}=0, \quad \therefore \frac{dy}{dx}=-\frac{b^2x}{a^2y}.$$

25. $y^2=4ax$, $dy/dx=(a/x)^{\frac{1}{2}}=2a/y$.
26. $x^2y+b^2x-a^2y=0$, $dy/dx=(b^2+2xy)/(a^2-x^2)$.
27. $x^3+y^3=3axy$, $dy/dx=-(x^2-ay)/(y^2-ax)$.
28. $x^2y-xy^2=a^3$, $dy/dx=(y^2-2xy)/(x^2-2xy)$.
29. If $y=\dfrac{1}{x}$, show that $\dfrac{dx}{\sqrt{1+x^4}}+\dfrac{dy}{\sqrt{1+y^4}}=0$.

CHAPTER IV.

TANGENTS AND NORMALS.

32. Let P and Q be two points near one another on a curve of which the equation is given. Let the coördinates of P be (x, y), then $x=OA$, $y=AP$. When x has the in-

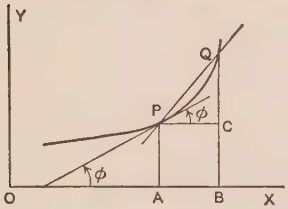

FIG. 13.

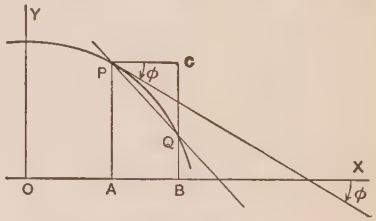

FIG. 14.

crement Δx or AB, the new value of y is BQ, hence CQ is Δy. Let the tangent at P make an angle ϕ with the x-axis. Then as in § 26, when Q approaches P as a limit of position, $\Delta x \doteq 0$, and

$$\tan \phi = \pounds \tan CPQ = \pounds(\Delta y/\Delta x) = dy/dx.$$

Let the length of the curve measured from some point up to P be s, and let the length of the arc PQ be Δs, and the length of the chord PQ be q. Then

$$\cos \phi = \pounds \cos CPQ = \pounds(\Delta x/q) = \pounds(\Delta x/\Delta s) \quad (\S\ 17) \quad = dx/ds.$$

Similarly, $$\sin \phi = dy/ds.$$

Thus, $$\cos \phi = \frac{dx}{ds} \ (1), \quad \sin \phi = \frac{dy}{ds} \ (2), \quad \tan \phi = \frac{dy}{dx} \ (3).$$

Squaring (1) and (2) and adding,

$$1=\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2,$$

or * $$ds^2=dx^2+dy^2.$$

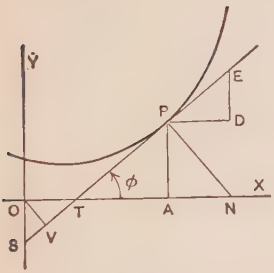

Fig. 15.

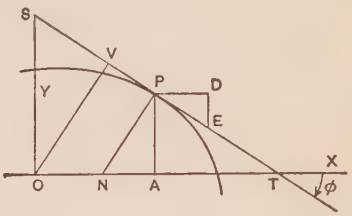

Fig. 16.

These relations show that if dx is PD (Figs. 15, 16), dy is DE, and ds is PE.

33. The *subtangent* $TA=y\dfrac{dx}{dy}$, the *tangent* † $TP=y\dfrac{ds}{dy}$,

the *subnormal* $AN=y\dfrac{dy}{dx}$, the *normal* $NP=y\dfrac{ds}{dx}$.

The intercepts of the tangent on the axes are

$$OT=x-y\frac{dx}{dy},\quad OS=-OT\tan\phi=y-x\frac{dy}{dx}.$$

Also, $$y-y_1=\left(\frac{dy}{dx}\right)_1(x-x_1)$$

is the equation of the tangent, and

$$y-y_1=-\left(\frac{dx}{dy}\right)_1(x-x_1)$$

---

* Powers of a differential dx are written dx^2, dx^3, etc. They must be distinguished from $d(x^2)$, $d(x^3)$, etc., which are differentials of powers of x.

† The line-segments known as the tangent and normal are the portions of the tangent and normal which join the x-axis to the point of contact.

the equation of the normal, at a point whose coördinates are (x_1, y_1), the parentheses $(\ )_1$ indicating the particular value which the enclosed quantity has when x_1 and y_1 are substituted for x and y.

It will be convenient to take dx as $+$, i.e., measured in the $+$ direction of the x-axis, and to suppose ϕ to be a positive or a negative acute angle; hence $\cos\phi$ and ds are always $+$, and $\sin\phi$ and $\tan\phi$ have the same sign as dy. It should be noticed that the curve rises or falls (y increases or decreases) according as dy is $+$ or $-$.

EXAMPLES.

1. The curve $a^2y = x(x^2 - a^2)$, Fig. 17.
Differentiating we have $\tan\phi = dy/dx = (3x^2 - a^2)/a^2$.
At the origin $x = 0$, $\therefore \tan\phi = -1$ and $\therefore \phi = -45°$.
At A or B, $x = \pm a$, $\therefore \tan\phi = 2$ and $\phi = 63° 26'$.
When $x = \pm a/\sqrt{3}$, $\tan\phi = 0$, $\therefore \phi = 0$.

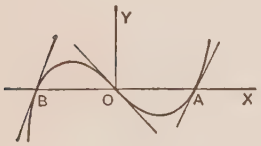

FIG. 17.

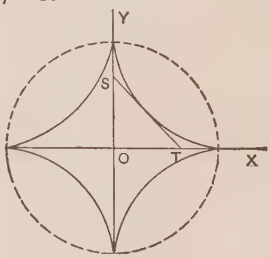

FIG. 18.

The equation of the tangent at any point (x_1, y_1) is

$$y - y_1 = \left(\frac{3x_1^2 - a^2}{a^2}\right)(x - x_1).$$

2. The common parabola $y^2 = 4ax$.
Differentiating each term, $2y\,dy = 4a\,dx$, $\therefore dy/dx = 2a/y$.
$\therefore y - y_1 = (2a/y_1)(x - x_1)$ is the equation of the tangent at (x_1, y_1), and reduces to $y_1y = 2a(x + x_1)$. The subnormal $= y\,dy/dx = 2a$, a constant.

3. The equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ represents an astroid, or four-cusped hypocycloid (Fig. 18), i.e., the locus of a point in the circum-

ference of a circle which rolls inside the circumference of a fixed circle, the diameter of the latter being four times that of the former. Differentiating the equation, we get $\frac{2}{3}x^{-\frac{1}{3}}\,dx+\frac{2}{3}y^{-\frac{1}{3}}\,dy=0$, whence $dx/dy=-(x/y)^{\frac{1}{3}}$. The intercepts of the tangent on the axes will be found to be $a^{\frac{2}{3}}\,x^{\frac{1}{3}}$ and $a^{\frac{1}{3}}\,y^{\frac{1}{3}}$. Squaring, adding, and taking the square root we find that the part of the tangent intercepted between the axes is of constant length, viz., a. Hence, if a straight line of length a slide with its extremities on two given lines at right angles to one another, it will constantly touch this curve.

4. To find $\tan\phi$ at any point of the curve $x^2y-xy^2=2$.

Differentiating each term by (C),

$$x^2\,dy+2xy\,dx-2xy\,dy-y^2\,dx=0,$$
$$\therefore\ dy/dx=(y^2-2xy)/(x^2-2xy).$$

5. Find the equations of the tangents at the points $(-1,\ 1)$, $(2,\ 1)$ on this curve. *Ans.* $x-y+2=0$, $x=2$.

6. Of the rectangular hyperbola $xy=k^2$ show that

(1) the equation of the tangent at $(x_1,\ y_1)$ is $x/x_1+y/y_1=2$,

(2) the equation of the normal at $(k,\ k)$ is $y=x$,

(3) the subtangent always $=-$the abscissa,

(4) the tangent makes with the axes a triangle of constant area, viz., $2k^2$.

7. Show that the tangent to the curve $(x+a)^2y=a^2x$ is parallel to the axis of x when $x=a$, perpendicular to it when $x=-a$, and that the tangent at the origin bisects the angle between the axes.

8. Find the equations of the tangent and normal at the point $(a,\ a)$ on the curve $ay^2=x^3$. *Ans.* $3x-2y=a$, $2x+3y=5a$.

Also show that the subtangent $=\frac{2}{3}a$, the subnormal $=\frac{3}{2}a$, the tangent $=\frac{1}{3}a\sqrt{13}$, the normal $=\frac{1}{2}a\sqrt{13}$.

9. On the curve $x^2y+b^2x=a^2y$, show that $\tan\phi=b^2/a^2$ when $x=0$, $20b^2/9a^2$ when $x=\frac{1}{2}a$, and $5b^2/9a^2$ when $x=2a$.

10. Show that the curves $y(4+x^2)=8$, $4y=x^2$, intersect at an angle $\tan^{-1}3$.

11. Find the equations of the tangents of the following curves at the given points:

(1) $xy=1+x^3$ at $(1,\ 2)$. *Ans.* $x-y+1=0$.

(2) $x^2+y^2=x^3$ at $(2,\ 2)$. $2x-y=2$.

(3) $x^n+y^n=x^{n+1}$ at $(2,\ 2)$. $(n+2)x-ny=4$.

(4) $a^2y=x^3$ at (x_1, y_1). $3x_1^2x-a^2y=2x_1^3$.

(5) $y^2=3x+1$ at (x_1, y_1). $3x-2y_1y+3x_1+2=0$.

(6) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1). $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$.

12. $4xy=4+x^3$; show that at $(2, \frac{3}{2})$ the subnormal $=\frac{3}{8}$, the subtangent $=2$, the normal $=\frac{17}{8}$, the tangent $=\frac{5}{2}$.

13. $y^2=3x+1$; show that when $y=-4$ the subnormal $=\frac{8}{3}$, the subtangent $=\frac{32}{3}$, the normal $=-\frac{1}{2}\sqrt{73}$, the tangent $=\frac{4}{3}\sqrt{73}$.

14. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, Fig. 18; show that $ds=(a/x)^{\frac{1}{3}}dx$.

15. Find an expression for the length of the perpendicular from the origin on the tangent at any point (x, y) of any curve.

Ans. $(x\,dy-y\,dx)/ds$.

16. In the case of the parabola $y^2=4ax$, show that the length of this perpendicular $=x\sqrt{a/(a+x)}$.

CHAPTER V.

DIFFERENTIALS OF EXPONENTIALS AND LOGARITHMS.

34. Differentials of the exponentials a^v and e^v. Let a be any constant, v any function of x. Then

$$\begin{aligned}\Delta(a^v) &= a^{v+\Delta v} - a^v = a^v(a^{\Delta v} - 1)\\ &= a^v[A\,\Delta v + \tfrac{1}{2}A^2(\Delta v)^2 + \dots],\quad A = \log_e a,\end{aligned}$$

by the exponential theorem, the series being convergent for all values of Δv.

$$\therefore\ \frac{\Delta(a^v)}{\Delta x} = Aa^v\,\frac{\Delta v}{\Delta x}(1 + \tfrac{1}{2}A\,\Delta v + \dots),$$

and taking the limits,

$$\frac{d(a^v)}{dx} = Aa^v\frac{dv}{dx},$$

or

$$d(a^v) = Aa^v\,dv. \qquad \text{(E)}$$

When $a = e = 2.71828\dots$, A is 1. Hence

$$d(e^v) = e^v\,dv. \qquad \text{(F)}$$

Ex. 1. $d(e^{3x}) = e^{3x}\,d(3x) = 3e^{3x}\,dx$.
2. $d(e^{-x}) = e^{-x}\,d(-x) = -e^{-x}\,dx$.
3. $d(2^{-x}) = (\log_e 2)2^{-x}\,d(-x) = -(\log_e 2)2^{-x}\,dx$.

35. Differentials of logarithms. First suppose the logarithms to be Napierian (or hyperbolic or natural) logarithms, the base being $e = 2{\cdot}71828\dots$ Let $y = \log_e v$,

then $$v = e^v, \quad \therefore\ dv = e^v\, dy = v\, dy,$$

$$\therefore\ dy, \quad \text{or} \quad d(\log_e v) = \frac{dv}{v}. \tag{G}$$

Secondly, let the base be any number a. Then $\because\ \log_a v = M \log_e v$, where $M = 1/\log_e a$, or $= \log_a e$,

$$\therefore\ d(\log_a v) = M \frac{dv}{v}. \tag{G$_1$}$$

M is the modulus of the system of logarithms with base a.

Unless the contrary is indicated, *the logarithms are always assumed to be Napierian.*

Ex. 1. $d \log (ax^3) = d(ax^3)/ax^3 = 3\ ax^2\ dx/ax^3 = 3\ dx/x$.
2. $d(x \log x) = \log x\, dx + x\, d(\log x) = (\log x + 1)dx$.

36. To differentiate u^v, where both u and v are variable quantities. Let $y = u^v$, then $\log y = (\log u)v$; hence, differentiating,

$$\frac{dy}{y} = v \frac{du}{u} + (\log u)dv, \quad \therefore\ dy = y\left[v \frac{du}{u} + (\log u)\, dv\right]$$

$$\therefore\ d(u^v) = v\, u^{v-1}\, du + (\log u)u^v\, dv, \tag{G$_2$}$$

i.e., the differential is obtained by supposing u and v in turn to vary while the other remains constant, and adding the results.

37. When an expression is made up of factors it is often simpler to take logarithms before differentiating.

Ex. $$y = (x+1)^{\frac{1}{2}}(x+3)^{\frac{9}{2}}/(x+4)^4,$$

$$\log y = \tfrac{1}{2} \log (x+1) + \tfrac{9}{2} \log (x+3) - 4 \log (x+4),$$

$$\frac{dy}{y} = \frac{1}{2}\frac{dx}{x+1} + \frac{9}{2}\frac{dx}{x+3} - 4\frac{dx}{x+4},$$

whence dy.

Examples.

1. $y=\frac{1}{2a}\log\frac{x-a}{x+a}=\frac{1}{2a}[\log(x-a)-\log(x+a)]$, $dy=\frac{dx}{x^2-a^2}$.
2. $y=\log(x/\sqrt{1+x^2})=\log x-\frac{1}{2}\log(1+x^2)$, $dy=dx/(x+x^3)$.
3. $f(x)=\sqrt{x}-\log(1+\sqrt{x})$, $f'(x)=\frac{1}{2}(1+\sqrt{x})^{-1}$.
4. $d\log(x+\sqrt{x^2\pm a^2})=dx/\sqrt{x^2\pm a^2}$.
5. $y=\log[e^x/(1+e^x)]$, $dy=dx/(1+e^x)$.
6. $y=e^{x^n}$, $dy=ne^{x^n}x^{n-1}\,dx$.
7. $d\log(\sqrt{x-a}+\sqrt{x-b})=\frac{1}{2}dx/\sqrt{(x-a)(x-b)}$.
8. $y=a^{\log x}=(e^A)^{\log x}=(e^{\log x})^A=x^A$, $dy=A\,x^{A-1}\,dx$.
9. $d(x^x)=(1+\log x)x^x\,dx$.
10. $y=1+xe^y$, $dx=(2-y)\,dy/e^y$.
11. $y=\log(y/x)$, $(1-y)x\,dy=y\,dx$.

12. Differentiate $y=uv$, $y=uvw$, and $y=u/v$, after taking logarithms, and compare the results with formulæ (C), (C_1), and (D).

13. Differentiate $y=v^n$ after taking logarithms, thus showing that $d(v^n)=nv^{n-1}\,dv$.

14. Show that the subtangent of the exponential curve $y=a^x$ is constant and $=\log_a e$.

15. Find the subnormal of the curve $y^2=a^2\log x$.

Ans. $a^2/2x$.

CHAPTER VI.

DIFFERENTIALS OF DIRECT CIRCULAR (TRIGONOMETRICAL) FUNCTIONS.

38. To differentiate sin v. Suppose v to be a function of θ. Then $\Delta \sin v = \sin(v+\Delta v) - \sin v = 2\cos(v+\frac{1}{2}\Delta v)\sin\frac{1}{2}\Delta v$,*

$$\therefore \pounds\frac{\Delta \sin v}{\Delta\theta} = \pounds\frac{2\cos v \,.\, \frac{1}{2}\Delta v}{\Delta\theta}, \quad (\S 17)$$

or
$$\frac{d\sin v}{d\theta} = \cos v \frac{dv}{d\theta}.$$

$$\therefore d\sin v = \cos v\, dv. \tag{1}$$

Similarly, $$d\cos v = -\sin v\, dv. \tag{2}$$

39. The differentials of the remaining functions may be found by first expressing them in terms of sine and cosine The results with (1) and (2) above are: †

$$d\sin v = \cos v\, dv, \tag{H}$$

$$d\cos v = -\sin v\, dv, \tag{I}$$

$$d\tan v = \sec^2 v\, dv, \tag{J}$$

$$d\cot v = -\operatorname{cosec}^2 v\, dv, \tag{K}$$

$$d\sec v = \sec v \tan v\, dv, \tag{L}$$

$$d\operatorname{cosec} v = -\operatorname{cosec} v \cot v\, dv. \tag{M}$$

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* Since $\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$.

† To these results may be added:

$d\,\text{vers}\, v = d(1-\cos v) = \sin v\, dv$,

$d\,\text{covers}\, v = d(1-\sin v) = -\cos v\, dv$.

Examples.

1. $d \sin n\theta = \cos n\theta \, d(n\theta) = n \cos n\theta \, d\theta$.
2. $d \sin (\tan \theta) = \cos (\tan \theta) \, d(\tan \theta) = \cos (\tan \theta) \sec^2\theta \, d\theta$.
3. $y = \tan \theta - \theta, \quad dy = \tan^2\theta \, d\theta$.
4. $d(\frac{1}{2}\theta - \frac{1}{4} \sin 2\theta) = \sin^2\theta \, d\theta$.
5. $d(\frac{1}{2}\theta + \frac{1}{4} \sin 2\theta) = \cos^2\theta \, d\theta$.
6. $d(\sec \theta + \tan \theta) = (1 + \sin \theta) \, d\theta / \cos^2 \theta$.
7. $y = \frac{1}{3} \tan^3 x - \tan x + x, \quad dy = \tan^4 x \, dx$.
8. $f(x) = \sin x - \frac{1}{3} \sin^3 x, \quad f'(x) = \cos^3 x$.
9. $d(\sin^2 x \cos^2 x) = \frac{1}{2} \sin 4x \, dx$.
10. $y = \log \tan \frac{1}{2}\theta, \quad dy = \operatorname{cosec} \theta \, d\theta$.
11. $y = \log \tan (\frac{1}{4}\pi + \frac{1}{2}\theta), \quad dy = \sec \theta \, d\theta$.
12. $d(\sec \theta + \log \tan \frac{1}{2}\theta) = \sec^2\theta \operatorname{cosec} \theta \, d\theta$.
13. $d\left(\dfrac{\sin x}{1 + \tan x}\right) = \dfrac{(\cos^3 x - \sin^3 x) dx}{(\sin x + \cos x)^2}$.
14. $f(x) = \sin (\log x), \quad f'(x) = x^{-1} \cos (\log x)$.
15. $de^x \cos x = e^x \, (\cos x - \sin x) \, dx$.
16. $d \log \sin \theta = \cot \theta \, d\theta$.
17. $d \log \cos \theta = -\tan \theta \, d\theta$.
18. $d \log \tan \theta = \sec \theta \operatorname{cosec} \theta \, d\theta$.
19. $d \log \sec \theta = \tan \theta \, d\theta$.
20. $d \log (\sec \theta + \tan \theta) = \sec \theta \, d\theta$.
21. $y = \log \sqrt{\sin x} + \log \sqrt{\cos x}, \quad dy/dx = \cot 2x$.
22. $y = 2/(1 + \tan \frac{1}{2}x), \quad dy/dx = -1/(1 + \sin x)$.
23. $d \log \sqrt{(1 - \cos \theta)/(1 + \cos \theta)} = \operatorname{cosec} \theta \, d\theta$.
24. $d \sin n\theta \sin^n \theta = n \sin^{n-1}\theta \sin (n+1) \, \theta \, d\theta$.
25. By differentiating $\sin 2\theta = 2 \sin \theta \cos \theta$, show that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

26. **The cycloid.** This is the curve traced by a point in the circumference of a circle which rolls along a straight line, Fig. 19.

Let θ = the angle through which the circle (of radius a) rolls while the tracing-point moves from O to P.

Then

$$x = OM = OB - MB = \text{arc } PB - PD = a\theta - a \sin \theta,$$
$$y = MP = BC - DC = a - a \cos \theta.$$

From these two results θ may be eliminated; but as the result-

ing equation is not algebraical we shall suppose the locus determined by the simultaneous equations

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

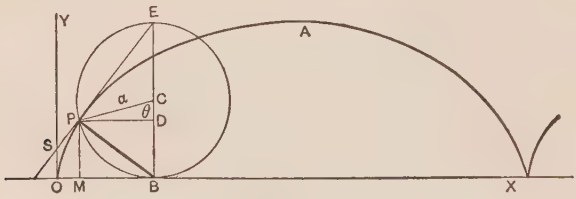

Fig. 19.

For a single arch of the curve θ varies from 0 to 2π; for greater or smaller values of θ the curve is repeated indefinitely in both directions.

Produce BC to meet the circle in E, then PE is the tangent and PB the normal at P. For

$$dx = a(1 - \cos\theta)\, d\theta = BD\, d\theta, \quad dy = a \sin\theta\, d\theta = DP\, d\theta,$$

$\therefore$ if the tangent makes an angle ϕ with the axis of x,

$$\tan\phi = dy/dx = DP : BD = \tan DBP = \tan DPE,$$

BPE, an angle in a semicircle, being a right angle. Therefore PE is the tangent. Hence at each instant the circle may be supposed

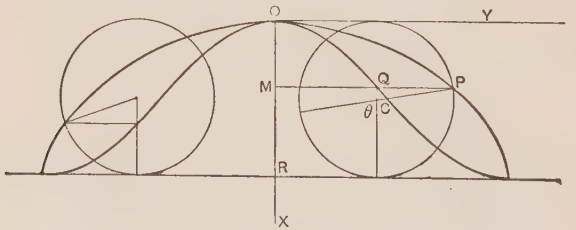

Fig. 20.

to be turning about its lowest point as an instantaneous centre of rotation.

Since $CE = CP$, $CEP = \frac{1}{2}\theta$, $\therefore$ the normal $PB = 2a \sin\frac{1}{2}\theta$.

27. If the axes of the cycloid be taken as in Fig. 20, its equations are

$$x = a(1 - \cos\theta), \quad y = a(\theta + \sin\theta),$$

θ being the angle through which the circle has rolled from R.

The locus of Q (of D in Fig. 19) is called the "companion to the cycloid." Its equations are

$$x = a(1 - \cos\theta), \quad y = a\theta.$$

Show that in this curve $\tan\phi = \operatorname{cosec}\theta$, and hence that ϕ is least when $x = a$.

28. At any point of the cycloid, show that

$$\frac{dy}{dx} = \sqrt{\frac{2a}{y} - 1} = \frac{\sqrt{2ay - y^2}}{y}, \quad \text{Fig. 19,}$$

$$\frac{dy}{dx} = \sqrt{\frac{2a}{x} - 1} = \frac{\sqrt{2ax - x^2}}{x}, \quad \text{Fig. 20.}$$

29. In the cycloid, Fig. 20, show that

$$ds/dx = \sqrt{(2a)/x}.$$

CHAPTER VII.

DIFFERENTIALS OF INVERSE CIRCULAR (TRIGONOMETRICAL) FUNCTIONS.

40. To differentiate $\sin^{-1}\frac{v}{a}$, the radian measure of the angle whose sine is v/a, a being a constant.

Let $y=\sin^{-1}\frac{v}{a}$, then $v=a\sin y$.

$\therefore\ dv=a\cos y\,dy=a\sqrt{1-\frac{v^2}{a^2}}\,dy=\sqrt{a^2-v^2}\,dy.\quad\therefore\ dy$ or

$d\sin^{-1}\frac{v}{a}=\frac{dv}{\sqrt{a^2-v^2}}$ (N), or $d\sin^{-1}v=\frac{dv}{\sqrt{1-v^2}}$, if $a=1$.*

Similarly,

$d\cos^{-1}\frac{v}{a}=-\frac{dv}{\sqrt{a^2-v^2}}$ (N_1), or $d\cos^{-1}v=-\frac{dv}{\sqrt{1-v^2}}$, if $a=1$;

$d\tan^{-1}\frac{v}{a}=\frac{a\,dv}{a^2+v^2}$ (P), or $d\tan^{-1}v=\frac{dv}{1+v^2}$, if $a=1$;

$d\cot^{-1}\frac{v}{a}=-\frac{a\,dv}{a^2+v^2}$ (P_1), or $d\cot^{-1}v=-\frac{dv}{1+v^2}$, if $a=1$;

$d\sec^{-1}\frac{v}{a}=\frac{a\,dv}{v\sqrt{v^2-a^2}}$ (Q), or $d\sec^{-1}v=\frac{dv}{v\sqrt{v^2-1}}$, if $a=1$;

$d\operatorname{cosec}^{-1}\frac{v}{a}=-\frac{a\,dv}{v\sqrt{v^2-a^2}}$ (Q_1), or $d\operatorname{cosec}^{-1}v=-\frac{dv}{v\sqrt{v^2-1}}$, if $a=1$.

* This formula should be preceded by a minus sign if $\cos y$ is $-$ i.e., if the angle is a second or third quadrant angle (see Fig. 7). The formulæ as given may be supposed to apply only to first-quadrant angles

EXAMPLES.

1. $d \sin^{-1}(2x^2) = \dfrac{d(2x^2)}{\sqrt{1-(2x^2)^2}} = \dfrac{4x\,dx}{\sqrt{1-4x^4}}$.

2. $d(\log \tan^{-1}x) = \dfrac{d \tan^{-1}x}{\tan^{-1}x} = \dfrac{dx}{(1+x^2)\tan^{-1}x}$.

3. $d \sin^{-1} 3x^2 = \dfrac{6x\,dx}{\sqrt{1-9x^4}}$.

4. $d \cos^{-1}\dfrac{a}{x} = \dfrac{a\,dx}{x\sqrt{x^2-a^2}}$.

5. $d \sin^{-1}\dfrac{x-a}{x} = \dfrac{a\,dx}{x\sqrt{2ax-a^2}}$.

6. $d \cos^{-1}\dfrac{a-x}{a} = \dfrac{dx}{\sqrt{2ax-x^2}}$.

7. $d \sin^{-1}\dfrac{x^2}{a^2} = \dfrac{2x\,dx}{\sqrt{a^4-x^4}}$.

8. $d \tan^{-1}e^x = \dfrac{dx}{e^x+e^{-x}}$.

9. $d \sec^{-1}\sqrt{1+x^2} = dx/(1+x^2)$.

10. $d \tan^{-1}(\sqrt{1+x^2}-x) = -\frac{1}{2}dx/(1+x^2)$.

11. $d \sin^{-1}(x/\sqrt{1+x^2}) = dx/(1+x^2)$.

12. $d \sin^{-1}[(1-x^2)/(1+x^2)] = -2dx/(1+x^2)$.

13. $d \tan^{-1}[2x/(1-x^2)] = 2dx/(1+x^2)$.

14. $d \sin^{-1}\sqrt{\sin x} = \frac{1}{2}\sqrt{1+\operatorname{cosec} x}\,dx$.

15. $d \operatorname{vers}^{-1}\dfrac{x}{a} = d \cos^{-1}\left(1-\dfrac{x}{a}\right) = \dfrac{dx}{\sqrt{2ax-x^2}}$.

16. $d \sin^{-1}\sqrt{\dfrac{x-a}{b-a}} = \dfrac{dx}{2\sqrt{(x-a)(b-x)}}$.

17. $y = a \sin^{-1}(x/a) + \sqrt{a^2-x^2}$, $dy = dx\sqrt{(a-x)/(a+x)}$.

18. $y = \sqrt{x^2-a^2} - a \sec^{-1}(x/a)$, $dy = dx\sqrt{x^2-a^2}/x$.

19. $y = x \tan^{-1}x - \log\sqrt{1+x^2}$, $dy = \tan^{-1}x\,dx$.

20. $y = \tan^{-1}x + \log\dfrac{x}{\sqrt{1+x^2}}$, $dy = \dfrac{(1+x)\,dx}{x(1+x^2)}$.

21. $y = \log\sqrt{\dfrac{x-a}{x+a}} + \tan^{-1}\dfrac{x}{a}$, $\dfrac{dy}{dx} = \dfrac{2ax^2}{x^4-a^4}$.

CHAPTER VIII.

DIFFERENTIALS OF HYPERBOLIC FUNCTIONS.

41. Def. The quantities $\frac{1}{2}(e^x - e^{-x})$, $\frac{1}{2}(e^x + e^{-x})$ are called,

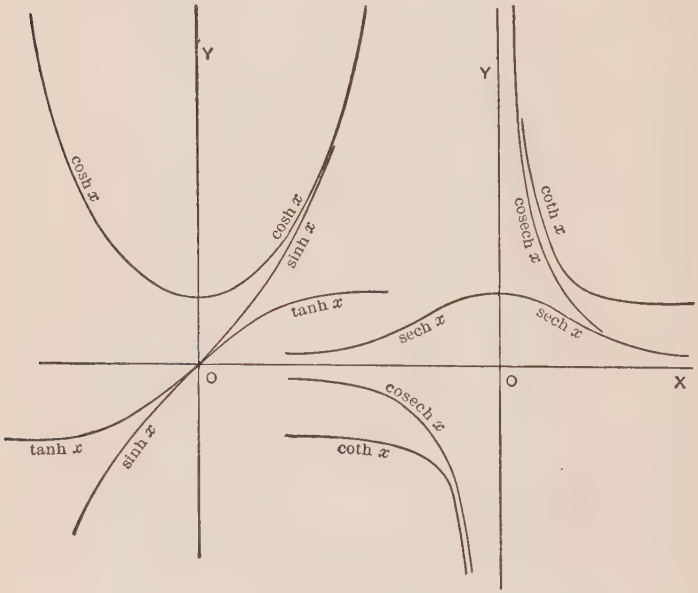

FIG. 21.

respectively, the hyperbolic sine (sinh x *) and hyperbolic cosine (cosh x) of x. The hyperbolic tangent of x is defined

* This may be read "sine h of x."

to be $\sinh x/\cosh x$, and the hyperbolic secant, cosecant, and cotangent to be the reciprocals of the cosine, sine, and tangent, respectively.

The graphs of the functions are represented in Fig. 21. Observe that $\sinh x$ may have any value, $\cosh x \geqq 1$, $\tanh x > -1$ and < 1, $\coth x > 1$ or < -1, etc., $\sinh 0 = 0$, $\cosh 0 = 1$, etc.

The fundamental relations

$$\cosh^2 x - \sinh^2 x = 1, \quad \operatorname{sech}^2 x = 1 - \tanh^2 x,$$
$$\operatorname{cosech}^2 x = \coth^2 x - 1,$$

are easily verified.

The differentials of the hyperbolic functions are similar to those of the circular functions. Only the most important are given here. (For the others see Appendix, Note C.)

Differentiating $\sinh v = \frac{1}{2}(e^v - e^{-v})$, $\cosh v = \frac{1}{2}(e^v + e^{-v})$, we have

$$d \sinh v = \cosh v \, dv,$$
$$d \cosh v = \sinh v \, dv,$$

whence may be deduced

$$d \sinh^{-1} \frac{v}{a} = \frac{dv}{\sqrt{v^2 + a^2}},$$

$$d \cosh^{-1} \frac{v}{a} = \frac{dv}{\sqrt{v^2 - a^2}},$$

$$d \tanh^{-1} \frac{v}{a} = \frac{a \, dv}{a^2 - v^2}, \quad v| < |a,$$

$$d \coth^{-1} \frac{v}{a} = \frac{a \, dv}{a^2 - v^2}, \quad v| > |a.$$

EXAMPLES.

1. $y = \log \cosh x, \quad dy/dx = \tanh x$

2. $y = \sec^{-1} \frac{x}{a} + \cosh^{-1} \frac{x}{a}, \quad dy = \frac{dx}{x} \sqrt{\frac{x+a}{x-a}}.$

3. $y=x\sqrt{a^2+x^2}+a^2\sinh^{-1}(x/a)$, $dy/dx=2\sqrt{a^2+x^2}$.

4. Show that $\sinh^{-1}\dfrac{x}{a}=\log\left(\dfrac{x+\sqrt{x^2+a^2}}{a}\right)$,

$$\tanh^{-1}\frac{x}{a}=\frac{1}{2}\log\left(\frac{a+x}{a-x}\right).$$

[Let $\sinh^{-1}x/a=z$ and $e^z=u$. Then $x/a=\sinh z=\frac{1}{2}(u-u^{-1})$. Solve for u in terms of x.]

5. **Gudermannian.** If $x=\log\tan(\frac{1}{4}\pi+\frac{1}{2}\theta)$, or $\log(\sec\theta+\tan\theta)$, θ is called the gudermannian of x ($\operatorname{gd}x$) and x is $\operatorname{gd}^{-1}\theta$.* Prove that

$$d\operatorname{gd}x=\operatorname{sech}x\,dx,$$
$$d\operatorname{gd}^{-1}x=\sec x\,dx.$$

[Differentiate, $\operatorname{gd}x=2\tan^{-1}e^x-\frac{1}{2}\pi$, and
$\operatorname{gd}^{-1}x=\log(\sec x+\tan x)$.]

6. If $x=\log(\sec\theta+\tan\theta)$, prove that

$$\cosh x=\sec\theta,\quad \sinh x=\tan\theta,\quad \tanh x=\sin\theta.$$

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* The inverse gudermannian $\operatorname{gd}^{-1}\theta$ is also written $\lambda(\theta)$, i.e.,
$\lambda(\theta)=\log\tan(\frac{1}{4}\pi+\frac{1}{2}\theta)=\log(\sec\theta+\tan\theta)$.

CHAPTER IX.

DIFFERENTIALS AS INFINITESIMALS.

42. Let y be a function of x, dx an increment of x, and suppose y and its derivative to be continuous from x to $x+dx$. Let Δy, dy, be the increment and differential of y corresponding to dx. Let dx become smaller and smaller and $\doteq 0$, then Δy and (in general) dy are also infinitesimals. Since

$$\pounds \frac{\Delta y}{dx}=\frac{dy}{dx}, \quad \therefore \frac{\Delta y}{dx}=\frac{dy}{dx}+i, \quad (\S\ 8)$$

where i is infinitesimal. Hence

$$\Delta y=dy+I, \qquad (1)$$

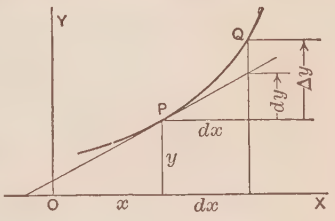

FIG. 22.

where I is an infinitesimal of an order higher than that of dx and dy.

If $dy \neq 0$, $I/dy \doteq 0$, and I becomes a very small part of dy. Hence dy, when very small, is a close approximation to Δy. In reality (1) implies that dy is what remains of Δy when the higher infinitesimals are omitted; in other words, if higher infinitesimals are left out of account dy may be used as if it were the increment of y corresponding to the increment dx of x.

If $y=f(x)$, (1) may be written

$$f(x+dx)-f(x)=f'(x)dx+I, \qquad (2)$$

where $I/dx \doteq 0$, and hence I is a very small part of dx when dx is very small.

43. Differentiation by the omission of the higher infinitesimals is much used in the practical applications of the subject, and may be illustrated by the following examples. It must be remembered that dx is now regarded as infinitesimal, and that the higher infinitesimals are not omitted because they are of trifling numerical value, but because they do not affect the final limit expressed by dy/dx. (See § 17.)

Ex. 1. If $y=x^n$, $\Delta y=(x+dx)^n-x^n$

$$=x^n+nx^{n-1}\,dx+\ldots-x^n=nx^{n-1}\,dx+\ldots,$$

the terms indicated by ... being higher infinitesimals. When these terms are omitted Δ changes into d.

$$\therefore\ d(x^n)=nx^{n-1}\,dx.$$

2. $y=e^x$, $\Delta y=e^{x+dx}-e^x=e^x(e^{dx}-1)=e^x(1+dx+\ldots-1)$.

$$\therefore\ dy=e^x\,dx.$$

3. $y=\sin x$, $\quad \Delta y=\sin(x+dx)-\sin x$

$$=\sin x\cos dx+\cos x\sin dx-\sin x.$$

But $\cos dx=1+I_1$, $\sin dx=dx+I_2$, (§ 16).

$$\therefore\ dy=\cos x\,dx.$$

4. To find the differential of the area A, Fig. 23.

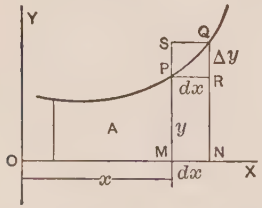

Fig. 23.

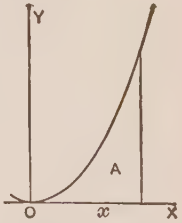

Fig. 24.

For the increment dx of x the increment ΔA of the area $=PMNQ$ $=PN+PRQ$. $PN=y\,dx$, and PRQ is a part of RS, which $=dx\,\Delta y$ and is $\therefore$ a higher infinitesimal.

$$\therefore\ dA=y\,dx.$$

For example, for the curve $y=x^2$, Fig. 24, $dA=x^2\,dx$, hence the relation connecting A and x is in this case $A=\frac{1}{3}x^3$.

5. **Barometric measurement of heights.** Let w_0 be the weight of a cubic inch of air at pressure p_0. Then the weight of a cubic inch at pressure p is (Boyle's law) w_0p/p_0, if the temperature is the same as before. Of a column of air of uniform temperature and one square inch in horizontal section consider the portion between sections at distances x, $x+dx$ from the top, and let p, $p+\Delta p$ be the pressures at top and bottom of this portion. Then Δp is the weight of this portion, dx its volume, its average pressure $>p$ and $<p+\Delta p$ and is therefore $p+i$, where i is infinitesimal. Hence

$$\Delta p=\frac{w_0(p+i)}{p_0}\,dx, \quad \therefore\ dp=\frac{w_0}{p_0}\,p\,dx, \quad \text{or} \quad dx=\frac{p_0}{w_0}\,\frac{dp}{p}.$$

This shows that the relation connecting x and p is

$$x=\frac{p_0}{w_0}\log p\ +\ c, \tag{1}$$

where c is a constant. Let the pressures at top and bottom of the whole column be P_1, P_2, and h the total depth. Then $p=P_1$ when $x=0$, and $p=P_2$ when $x=h$. Substituting in (1) and subtracting,

$$h=\frac{p_0}{w_0}(\log P_2-\log P_1)=\frac{p_0}{w_0}\log\left(\frac{P_2}{P_1}\right).$$

The values of the constants p_0 and w_0 are to be supplied from experiment.

Miscellaneous Examples.

1. $x^y=y^x$, show that $\dfrac{dy}{dx}=\dfrac{y^2(1-\log x)}{x^2(1-\log y)}$.

2. If $x=e^{\frac{x-y}{y}}$, $dy/dx=\log x/(1+\log x)^2$.

3. $d\left[\log\sqrt[4]{\dfrac{x-1}{x+1}}-\frac{1}{2}\tan^{-1}x\right]=\dfrac{dx}{x^4-1}$.

4. $d\cos^{-1}\left(\dfrac{b+a\cos x}{a+b\cos x}\right)=\dfrac{\sqrt{a^2-b^2}\,dx}{a+b\cos x}$.

5. Find the derivative of x with respect to $\tan x$.

Ans. $\cos^2 x$.

6. Find the derivative of $\sin^{-1}x$ with respect to $\sqrt{1-x^2}$.

Ans. $-x^{-1}$.

7. Differentiate $\tan x$ directly.*

8. Differentiate $\tan^{-1}x$ directly.†

9. If $f(x)=\log\left(\frac{1-x}{1+x}\right)$, show that $f(x)+f(y)=f\left(\frac{x+y}{1+xy}\right)$.

10. If $(x-a)^n$ is a factor of $f(x)$, show that $(x-a)^{n-1}$ is a factor of $f'(x)$. [Assume $f(x)=(x-a)^n F(x)$].

Hence explain a method of finding the equal roots of an algebraical equation.

11. If $f(x)$ contains a factor $(x-a)^{-1}$ which causes it to be infinite when $x=a$, show that $f'(x)$ is also infinite.

12. A function $f(x)$ is said to be an even function of its variable x if $f(-x)=f(x)$, and an odd function if $f(-x)=-f(x)$. What are the geometrical peculiarities of the graphs of such functions?

Show that the following are even functions:

$$\cos x, \quad x\sin x, \quad (e^x-e^{-x})/x, \quad x/(e^x-1)+\tfrac{1}{2}x.$$

Show that the following are odd functions:

$$x^3\sec x, \quad \tanh x, \quad \log(\sec x+\tan x).$$

13. Show that $\cosh^{-1}x$ and $\operatorname{sech}^{-1}x$ are double-valued functions.

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* $\tan A-\tan B=\sin(A-B)/\cos A\cos B$.

† $\tan^{-1}m-\tan^{-1}n=\tan^{-1}(m-n)/(1+mn)$.

CHAPTER X.

FUNCTIONS OF MORE THAN ONE VARIABLE.

44. Such functions may be differentiated by the formulæ already given.

Ex. 1. $u=(x+y^2)^3$. Here u is to be regarded as a function of x and y, both of which are assumed to have differentials. We have

$$du=3(x+y^2)^2\,d(x+y^2)=3(x+y^2)^2(dx+2y\,dy),$$

$$\therefore\ du=3(x+y^2)^2\,dx+6y(x+y^2)^2\,dy.$$

2. $u=\sin^{-1}\left(\frac{x}{y}\right)$, $\quad du=\dfrac{d\left(\frac{x}{y}\right)}{\sqrt{1-\left(\frac{x}{y}\right)^2}}$

$$=\frac{y\,dx-x\,dy}{y\sqrt{y^2-x^2}}=\frac{dx}{\sqrt{y^2-x^2}}-\frac{x\,dy}{y\sqrt{y^2-x^2}}.$$

45. Partial differentials and derivatives. It will be observed in these examples that the first term of the result is what we should have obtained if we had differentiated u on the supposition that x alone varied, y being regarded as a constant; let this be written $d_x u$. The second term is what we should have obtained if we had differentiated on the supposition that y alone varied, and this we call $d_y u$. Hence in these examples

$$du=d_x u+d_y u.$$

The same thing is true for all continuous functions of two variables. For, if we differentiate by the ordinary methods, we shall in every case get a result which may be written

$$du = M\,dx + N\,dy,$$

where M and N may contain x and y, but not dx or dy. If $dy = 0$, the right-hand side reduces to $M\,dx$, which is therefore $d_x u$, the differential of u on the supposition that y is constant (i.e., $dy = 0$) while x varies. Similarly $N\,dy$ is $d_y u$, the differential of u on the supposition that y alone varies.

$$\therefore\ du = d_x u + d_y u. \tag{1}$$

The differentials $d_x u$, $d_y u$ are called *partial differentials* of u with regard to x or y, du being called the *total differential* of u. The total differential is therefore equal to the sum of the partial differentials.

Similarly if u is a function of three variables x, y, z,

$$du = d_x u + d_y u + d_z u. \tag{2}$$

It should be noticed that formulæ (C), (C_1), (D), (G_2) are particular cases of functions of two or more variables.

The result (1) may be put into the form

$$du = \frac{d_x u}{dx}\,dx + \frac{d_y u}{dy}\,dy, \tag{3}$$

which brings out clearly the fact that the coefficients of dx and dy are the partial derivatives of u, which are equal to the partial differential quotients. The subscripts are usually omitted, (3) becoming

$$du = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy. \tag{4}$$

The symbol ∂ is frequently employed to express partial differential quotients or derivatives, (4) being written *

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy. \tag{5}$$

* $\partial u/\partial x$ may be read "partial du by dx."

Ex. 1. $u=\sin(x^2+xy)$. Differentiating, first regarding x only as variable, and afterwards regarding y as the variable,

$$\partial u/\partial x=\cos(x^2+xy)\,.\,(2x+y),\quad \partial u/\partial y=\cos(x^2+xy)\;\;x.$$

2. $u=x^3+y^3+z^3-3xyz$,

$$\partial u/\partial x=3(x^2-yz),\quad \partial u/\partial y=3(y^2-zx),\quad \partial u/\partial z=3(z^2-xy).$$

46. If $u=f(x,\ y)$, dx and dy, being differentials of the variables, are increments of x and y. If dx and dy are taken as infinitesimal increments, du is not the same as Δu, the infinitesimal increment of the function. Since it may be obtained by the ordinary rules of differentiating, du is Δu when the higher infinitesimals are omitted (§ 42), or

$$\Delta u=du+I.$$

Hence when dx and dy are very small du is a close approximation to Δu.

EXAMPLES.

1. $u=\sin(x^2-y^2)$, $\quad d_xu=2x\cos(x^2-y^2)\,dx,$

$$d_yu=-2y\cos(x^2-y^2)\,dy.$$

2. $u=(x-y)/(x+y)$, $\quad du=2(y\,dx-x\,dy)/(x+y)^2.$
3. $u=(ax^2+by^2+cz^2)^n$,

$$du=2nu^{\frac{n-1}{n}}(ax\,dx+by\,dy+cz\,dz).$$

4. If $\tan\theta=y/x$, $\quad (x^2+y^2)\,d\theta=x\,dy-y\,dx.$
5. $u=x^y$, $\ d_xu=y\,x^{y-1}\,dx$, $\ d_yu=(\log x)\,x^y\,dy.$

$$\therefore\ du=y\,x^{y-1}\,dx+(\log x)\,x^y\,dy,\text{ as in }(G_2).$$

6. $u=\log(e^x+e^y)$, $\ \partial u/\partial x+\partial u/\partial y=1.$
7. $u=\tan^{-1}(x/y)$, $\ \partial u/\partial x=y/(x^2+y^2)$, $\ \partial u/\partial y=-x/(x^2+y^2).$
8. $u=\log(\tan x+\tan y)$, $\ \sin 2x\dfrac{\partial u}{\partial x}+\sin 2y\dfrac{\partial u}{\partial y}=2.$
9. $u=\log_y x$, $\ ux\,\partial u/\partial x+y\,\partial u/\partial y=0.$

[Note. $\log_y x=\log_e x/\log_e y.$]

10. Given $x=r\cos\theta$, $y=r\sin\theta$, show that

$$dx^2+dy^2=dr^2+r^2\,d\theta^2,$$
$$x\,dy-y\,dx=r^2\,d\theta.$$

11. (1) If a function u consists of terms such as ax^py^q, and $p+q$ is the same number n in each of the terms, u is said to be homogeneous and of the degree n. Show that for such a function *

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

(2) If $u=f(v)$ and v is homogeneous as defined, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nv\,f'(v).$$

These propositions may obviously be extended to functions of three or more variables. Verify in the case of Exs. 1 and 3.

47. Tangent and normal. Let $f(x, y)=c$ (c constant) be the equation of a curve. The first member of the equation is a function of x and y; calling it u and differentiating the equation we have, by § 45 (5),

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0, \tag{1}$$

whence dy/dx, the slope of the tangent at (x, y), is $-\dfrac{\partial u}{\partial x}\Big/\dfrac{\partial u}{\partial y}$.

Let (x_1, y_1) be the point of contact of a tangent to the curve, (x, y) any other point on the tangent. Then $x-x_1$ and $y-y_1$ are proportional to dx and dy. Hence, from (1),

$$\left(\frac{\partial u}{\partial x}\right)_1(x-x_1)+\left(\frac{\partial u}{\partial y}\right)_1(y-y_1)=0 \tag{2}$$

is the equation of the tangent, and

$$\frac{x-x_1}{\left(\dfrac{\partial u}{\partial x}\right)_1}=\frac{y-y_1}{\left(\dfrac{\partial u}{\partial y}\right)_1} \tag{3}$$

the equation of the normal, at (x_1, y_1). These equations are often more convenient than those of § 33.

* A particular case of Euler's theorem on homogeneous functions (Ex. 1, § 234).

Ex. Find the equations of the tangent and normal at the point (a, a) on the curve

$$x^3+y^3-2a\,xy=0.$$

$$\partial u/\partial x=3x^2-2ay=3a^2-2a^2=a^2 \quad \text{for} \quad (a, a).$$
$$\partial u/\partial y=3y^2-2ax=3a^2-2a^2=a^2 \quad \text{for} \quad (a, a).$$

$\therefore$ the tangent is $a^2(x-a)+a^2(y-a)=0$, or $x+y=2a$,

and the normal is $\dfrac{x-a}{a^2}=\dfrac{y-a}{a^2}$, or $x=y$.

48. Centre of a conic. Let the general equation of a conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

or $u=0$. When referred to parallel axes through the point (x_1, y_1) the terms of the first degree are

$$2(ax_1+hy_1+g)x+2(hx_1+by_1+f)y,$$

or

$$\left(\frac{\partial u}{\partial x}\right)_1 x+\left(\frac{\partial u}{\partial y}\right)_1 y.$$

The new origin is therefore the centre if $\left(\dfrac{\partial u}{\partial x}\right)_1=0$ and $\left(\dfrac{\partial u}{\partial y}\right)_1=0$. Hence the centre of the conic is the intersection of $\partial u/\partial x=0$ and $\partial u/\partial y=0$. If the coördinates of the point thus found satisfy the given equation, the centre is on the conic, which therefore consists of a pair of straight lines.

EXAMPLES.

1. Find the equation of the tangent at any point of the curve $(x/a)^m+(y/b)^m=2$, and show that $x/a+y/b=2$ is the tangent at the point (a, b).

2. Show that the length of the perpendicular from the origin on the tangent at (x, y) to the curve $u=c$ is

$$\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\Big/\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}.$$

3. In the case of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, Fig. 18, show that this perpendicular $=\sqrt[3]{axy}$.

4. In the case of the parabola $(x/a)^{\frac{1}{2}}+(y/b)^{\frac{1}{2}}=1$, show that this perpendicular $=[abxy/(ax+by)]^{\frac{1}{2}}$.

5. Find the centres of the conics

(1) $x^2-4xy-2y^2+6y=2$, *Ans.* $(1, \frac{1}{2})$.

(2) $18x^2-8xy+3y^2+8x-6y-5=0$. $(0, 1)$.

6. Show that $3x^2+5xy-2y^2-x+5y=2$ represents a pair of straight lines.

CHAPTER XI.

SMALL DIFFERENCES

49. When the differentials of the variables of a function are small increments, the differential of the function is a close approximation to the increment of the function (§§ 42, 46).

EXAMPLES.

1. Given $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, find $\sin 30° 1'$.

Here the angle increases by a small amount and it is required to find the small increment in the sine.

We have $d \sin \theta = \cos \theta \, d\theta$; $\cos \theta = \frac{1}{2}\sqrt{3}$, $d\theta = 60/206265$ rdn., $\therefore d \sin \theta = \cdot 0002519$, $\therefore \sin 30° 1' = \cdot 5002519$, which is correct to the last decimal place.

2. How much must be added to $\log_{10} \sin 30°$ to get $\log_{10} \sin 30° 1'$?

We have $d \log \sin \theta = \cos \theta \, d\theta / \sin \theta = \cdot 0005038$, which is the increase of the Napierian log.; the increase of the common log. is obtained by multiplying by the modulus.

$$\therefore \cdot 0005038 \times \cdot 4342945 = \cdot 0002188$$

is the required increment.

The difference columns in the mathematical tables are found or verified in this way.

3. The radius of a right circular cone is 3 inches and the height is 4 inches; if the radius were ·006 in. more, and the height ·003 in. less, what would be the change in the volume?

The volume $v = \frac{1}{3}\pi r^2 h$, $\therefore dv = \frac{1}{3}\pi(2rh \, dr + r^2 dh)$

$$= \tfrac{1}{3}\pi(2 \times 3 \times 4 \times \cdot 006 - 3^2 \times \cdot 003) = \cdot 1225 \text{ cub. in.}$$

4. Assuming that the radius of an iron ball increases by ·000011 of its original length for each degree of temperature, what will

be the increase in volume of an iron ball of 8 in. radius when the temperature is raised 25 degrees?

The volume $v = \frac{4}{3}\pi r^3$, $\therefore\ dv = 4\pi r^2\, dr$

$$= 4\pi \times 8^2 \times 25 \times \cdot 000011 \times 8 = 1.77 \text{ cub. in.}$$

5. In a certain triangle, $b = 445$, $c = 606$, $A = 62^\circ\ 51'\ 33''$, whence a is calculated and found to be 565; it is then noticed that A should have been $62^\circ\ 53'\ 31''$; what is the correction to a?

The change in A is $1'\ 58'' = 118'' = 118/206265$ rdn.

Also, $a^2 = b^2 + c^2 - 2bc \cos A$; differentiating this, supposing b and c constant, we have

$$2a\, da = 2bc \sin A\, dA, \quad \therefore\ da = bc \sin A\, dA/a$$

$$= 445 \times 606 \times \sin 62^\circ\ 51'\ 33'' \times 118/206265 \times 565 = \cdot 243.$$

An approximate value of $\sin A$ is sufficient in this place.

6. Given $\log_e 900 = 6{\cdot}8024$, find $\log_e 901$.

Increase of $\log x = dx/x = 1/900$. *Ans.* $6{\cdot}8035$.

7. Given $\log_{10} 1000 = 3$, find $\log_{10} 1002$. *Ans.* $3{\cdot}00087$.

8. Find $\tan 45^\circ\ 1'$. *Ans.* $1{\cdot}00058$.

9. On account of the rotation of the earth the correction to the weight w of a body is $-w \cos^2\lambda/289$, where λ is the latitude. What is the change in this correction for one mile north of lati tude 45° N.? the radius of the earth being assumed to be 4000 miles. *Ans.* $w/(289 \times 4000)$.

10. Find the relation connecting small differences of t and δ in the equation

$$\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t,$$

ϕ and h being constant.

Differentiating and arranging the terms, we get

$$dt = \left(\frac{\tan \phi}{\sin t} - \frac{\tan \delta}{\tan t}\right) d\delta.$$

This is the "Equation of Equal Altitudes" in astronomy.

11. Find the relation connecting small differences of δ and A in the equation

$$\sin \delta = \sin \phi \sin h - \cos \phi \cos h \cos A,$$

ϕ and h being constant.

Ans. $dA = \cos \delta\, d\delta/(\cos \phi \cos h \sin A)$.

12. In any plane triangle

(1) $da = \cos C\, db + \cos B\, dc + b \sin C\, dA$,

(2) $\dfrac{da}{a} - \dfrac{db}{b} = \dfrac{dA}{\tan A} - \dfrac{dB}{\tan B}$.

13. The sides a and b of a right-angled triangle ABC ($C = 90°$) receive small corrections da and db; what is the change in the perpendicular p from the right angle on the hypothenuse?

$$Ans.\ \frac{dp}{p^3} = \frac{da}{a^3} + \frac{db}{b^3}, \quad \text{or} \quad dp = \cos^3 A\ da + \cos^3 B\ db.$$

50. Solution of equations by approximation. If a is a value of x, and h a small increment of x, then, § 42 (2),

$$f(a+h) - f(a) = f'(a)h, \text{ nearly.}$$

Let $f(x) = 0$ be an equation, a a quantity which is known (by trial or otherwise) to be an approximate value of a root of the equation, $a+h$ to be that root, where h is small compared with a. Then $f(a+h) = 0$. $\therefore\ h = -f(a)/f'(a)$, nearly. Hence if a be a first approximation to the root,

$$a - \frac{f(a)}{f'(a)} \tag{1}$$

will be a nearer approximation. If this new value be substituted for a in (1), a nearer approximation still will be obtained, and with this a closer approximation, and so on.

FIG. 25.

If Fig. 25 is the graph of $f(x)$, the roots of the equation $f(x) = 0$ are the intercepts of the curve on the x-axis. If OA is the true value of a root and OB the assumed value, the first corrected value is OC. For, if $OB = a$, $f(a) = BE$,

$f'(a) = \tan BCE$, $\therefore f(a)/f'(a) = CB$. If OC is next assumed, the second corrected value is OD.

EXAMPLES.

1. $x^3 - 4x - 2 = 0$. Here $f(x) = x^3 - 4x - 2$, and $f'(x) = 3x^2 - 4$. Since $f(2) = -2$, and $f(3) = 13$, there is a root of the equation lying between 2 and 3. Let us then assume 2 as the first approximation. Then

$$2 - \frac{f(2)}{f'(2)} = 2 - \frac{-2}{8} = 2{\cdot}25$$

is a second approximation. Again,

$$2{\cdot}25 - \frac{f(2{\cdot}25)}{f'(2{\cdot}25)} = 2{\cdot}25 - {\cdot}035 = 2{\cdot}215,$$

a third approximation.

Since $f(0) = -2$, and $f(-1) = 1$, there is a root lying between 0 and -1. Assume it to be $-{\cdot}5$. Then the next approximation is $-{\cdot}538$. Again, taking $-{\cdot}54$ as this second approximation we find the next to be $-{\cdot}5392$.

Similarly $x = -1{\cdot}676$ is the third root.

2. $x^3 + 2x - 13 = 0$.	*Ans.*	$2{\cdot}069$.
3. $x^3 - x^2 - 2 = 0$.		$1{\cdot}696$.
4. $x^5 = 34$.		$2{\cdot}024$.
5. $x^5 - 12x = 200$.		$2{\cdot}982$.
6. $e^x(1 + x^2) = 40$.		$2{\cdot}046$.
7. $x^x = 5$.		$2{\cdot}129$.
8. $x^4 - 12x^2 + 12x - 3 = 0$.		$2{\cdot}858$, $-3{\cdot}907$

CHAPTER XII.

MULTIPLE POINTS.

51. Tangents at the origin. Let it be required to find the line which touches the curve $x^3+y^3=3axy$, Fig. 28, at the origin.

Differentiating the equation we obtain

$$dy/dx=-(x^2-ay)/(y^2-ax),$$

which for the point (0, 0) assumes the form 0/0. The difficulty here met with is avoided by the method now to be explained.

52. If a curve passes through the origin, its equation can contain no constant term; let it be

$$a_1x+b_1y+a_2x^2+b_2xy+c_2y^2+\ldots=0.$$

For x and y substitute $r\cos\theta$ and $r\sin\theta$, where r is the length of a straight line drawn from the origin to the point (x, y) on the curve, and θ is the angle which this line makes with the x-axis. The equation becomes

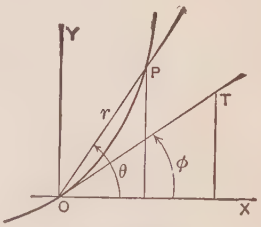

FIG. 26.

$$r(a_1\cos\theta+b_1\sin\theta)+r^2(a_2\cos^2\theta+\ldots)+\ldots=0.$$

One root is $r=0$, which implies that the curve passes through the origin; the remaining roots are given by the equation

$$a_1\cos\theta+b_1\sin\theta+r(a_2\cos^2\theta+\ldots)+\ldots=0.$$

Another root will $= 0$ if $a_1 \cos\theta + b_1 \sin\theta = 0$. Hence if the tangent at the origin makes an angle ϕ with the x-axis, ϕ is given by the equation

$$a_1 \cos\phi + b_1 \sin\phi = 0. \qquad (1)$$

If (x, y) is a point on the tangent at a distance t from the origin, $\cos\phi = x/t$ and $\sin\phi = y/t$. Substituting in (1) we have $a_1x + b_1y = 0$ for the equation of the tangent at the origin, i.e., the terms of the first degree in the given equation, equated to zero, represent the tangent at the origin.

If there are no terms of the first degree, it may be shown in the same way that $a_2x^2 + b_2xy + c_2y^2 = 0$ is the equation of a pair of tangents at the origin; and generally, *when the origin is a point on the curve, the terms of the lowest degree, equated to zero, represent the tangents at the origin.*

53. Multiple points. A point at which there are two or more tangents (i.e., where two or more branches of a curve intersect) is called a multiple point; it is called a *double* point, a *triple* point, etc., according as two, three, etc., branches intersect at the point.

When the equation $a_2x^2 + b_2xy + c_2y^2 = 0$ represents a pair of distinct lines the point is called a *node* (Figs. 27, 28).

When the lines are coincident the two branches of the curve touch one another and the tangent may be considered as a double tangent. Such a point is called a *cusp*, which is said to be of the first or second species according as the two branches of the curve lie on opposite sides (Figs. 29, 30) or on the same side (Fig. 31) of their common tangent; and to be double or single according as the branches lie on both sides (Fig. 34) or on one side only (Fig. 29) of their common normal. A cusp is also called a stationary point; for, considering the curve as the path of a moving point, at a cusp the point must come to rest and reverse its motion.

When the lines are imaginary the point is called a *conjugate point*. The coördinates of such a point satisfy the

equation of the curve, but the point is isolated from the rest of the locus which the equation represents.

EXAMPLES.

1. The lemniscate * $a^2(y^2-x^2)+(y^2+x^2)^2=0$, Fig. 27.

The origin is a node at which the tangents are $y^2-x^2=0$, i.e., $y=x$ and $y=-x$.

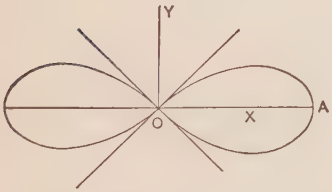

FIG. 27.

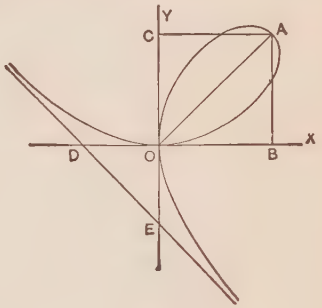

FIG. 28.

2. The folium † $x^3+y^3=3axy$, Fig. 28.

The origin is a node, the tangents being given by $xy=0$, i.e., $x=0$, $y=0$, the axes.

3. The semi-cubical parabola $ay^2=x^3$, Fig. 29.

The origin is a cusp, the tangents being given by $y^2=0$, i.e., two lines coinciding with the axis of x. Moreover, the curve is symmetrical with regard to the axis of x, and y is impossible if x is negative; hence the cusp is single and of the first species.

4. In the curve $(y-x)^2=x^3$, Fig. 30, the origin is a cusp at which the tangent is $y=x$; also, since $y-x=\pm x^{\frac{3}{2}}$, $y>x$ on one branch and $<x$ on the other, hence the cusp is of the first species.

---

* The curve is most easily plotted from its polar equation
$$r^2=a^2\cos 2\theta.$$

† This curve may be plotted as follows: Let $y=mx$ in the equation. Then $x=3am/(1+m^3)$ and $y=mx$ or $3am^2/(1+m^3)$. Thus x and y are expressed in terms of a third variable, and by giving arbitrary values to m the coördinates of any number of points on the curve may be calculated. The same substitution may be employed in other cases (e.g., Figs. 36, 37, 38) in which the equation contains terms of two degrees only. It should be noticed that m is the slope of the line drawn from the origin to the point (x, y) on the curve.

5. In the curve $(y-x^2)^2=x^5$, or $y=x^2(1\pm\sqrt{x})$, Fig. 31, the origin is a cusp, the tangent at which is $y=0$; also, y is + on both branches until $x=1$, and $\therefore$ the cusp is of the second species.

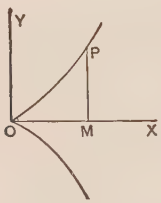

FIG. 29.

FIG. 30.

FIG. 31.

6. In the curve $y^2=x^2(2x+1)$, Fig. 32, the origin is a node at which the tangents are $y=\pm x$. But in the curve $y^2=x^2(2x-1)$, Fig. 33, the tangents are $y^2=-x^2$, and are $\therefore$ imaginary, and hence the origin is a conjugate point.

FIG. 32.

FIG. 33.

FIG. 34.

FIG. 35.

7 There are certain cases in which the origin is a conjugate point even when the terms of the second degree are a perfect square. Thus, in the curves $y^2=x^4(2x+1)$, Fig. 34, and $y^2=x^4(2x-1)$, Fig. 35, the origin is a double point and the tangents are given by $y^2=0$; in the first curve the origin is a double cusp, in the second a conjugate point, since y is imaginary for any value of x less than $\frac{1}{2}$.

8. The curve $ay^3-3ax^2y=x^4$, Fig. 36.

The origin is a triple point at which the tangents are $ay^3-3ax^2y=0$; i.e., $y=0$, $y=\pm x\sqrt{3}$.

9. In the curve $ay^4-ax^2y^2=x^5$, Fig. 37, the origin is a quadruple point, at which the tangents are $y=0$, $y=0$, $y=\pm x$.

10. $(x-y)^2=(x-1)^5$. The point (1, 1) is a cusp, for the equation referred to parallel axes through (1, 1) is $(x-y)^2=x^5$.

11. Find the tangents to the following curves at the origin:

(1) $(a^2+x^2)y^2=(a^2-x^2)x^2$. *Ans.* $y=\pm x$.
(2) $a^2y(x+y)=x^4$. $y=0,\ x+y=0$.
(3) $x(y-x)^2=y^4$. $x=0,\ y=x,\ y=x$.
(4) $a(y-x)(y^2+x^2)+x^4=0$. $y=x$.
(5) $y^3(y-x)=a(y^3+x^3)$. $x+y=0$.
(6) $(x-a)y=x(x-2a)$. $y=2x$.
(7) $y^2=(x-1)x^2$. Imaginary.

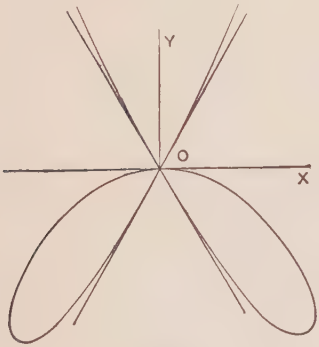

Fig. 36.

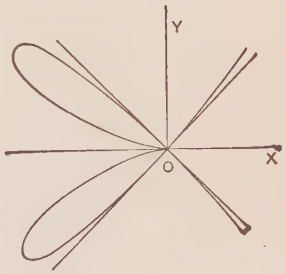

Fig. 37.

12. Show that the origin is a single cusp of the first species on the cissoid $y^2(a-x)=x^3$, Fig. 41.

13. Show that there is a node at the point (1, 2), on the curve $(y-2)^2=(x-1)^2x$.

14. Show that the point $(2a,\ 0)$ is a node on the curve $ay^2=(x-a)(x-2a)^2$.

15. Show that the point $(-a,\ 0)$ is a conjugate point on the curve $ay^2=x(a+x)^2$.

54. Let the equation of a curve, freed (if necessary) from fractions and radicals affecting the coördinates, be $f(x,\ y)=c$ or $u=c$. The tangent, § 47 (2), when referred to parallel axes through the point of contact $(x_1,\ y_1)$ is

$$\left(\frac{\partial u}{\partial x}\right)_1 x+\left(\frac{\partial u}{\partial y}\right)_1 y=0.$$

Hence if $\left(\frac{\partial u}{\partial x}\right)_1 = 0$ and $\left(\frac{\partial u}{\partial y}\right)_1 = 0$ the equation of the curve referred to the new axes will have no terms of the first degree. Conversely, points whose coördinates satisfy $\partial u/\partial x = 0$ and $\partial u/\partial y = 0$ as well as the given equation $u = c$ are multiple points of the curve.

Ex. 1. To examine the curve $(x-1)^5 - (2x-y)^2 = 0$ for multiple points.

$$\partial u/\partial x = 5(x-1)^4 - 4(2x-y) = 0,$$
$$\partial u/\partial y = 2(2x-y) = 0,$$

whence $x = 1$, $y = 2$. These coördinates satisfy the given equation, hence (1, 2) is a multiple point. Transforming to parallel axes through (1, 2) the equation becomes $x^5 - (2x-y)^2 = 0$, hence the point is a cusp at which the tangent is $2x = y$.

2. Examine the curve $x^3 - 2x^2 + y^2 - 4x + 2y + 9 = 0$ for multiple points. *Ans.* A conjugate point at (2, −1).

CHAPTER XIII.

ASYMPTOTES.

55. Definition. An *asymptote* of a curve is the limit of position of a secant when two of its points of intersection with the curve move away to an infinite distance, and hence also the limit of position of a tangent when the point of contact moves to an infinite distance.

56. Asymptotes by substitution.

Ex. 1. Of the curve $x^3+y^3=3axy$, Fig. 28, the line $y=mx+b$ is an asymptote if m and b are determined so that the line may meet the curve in two points infinitely distant. Substituting $mx+b$ for y in the equation of the curve we have

$$(1+m^3)x^3+3(m^2b-am)x^2+\ldots=0, \qquad (1)$$

the roots of which are the abscissas of the points of intersection of the line and the curve. Two of the roots become infinite * when m and b change so as to cause the coefficients of the two highest powers in (1) to $\doteq 0$. Hence the required values of m and b are obtained by solving the equations

$$1+m^3=0, \quad m^2b-am=0.$$

$\therefore m=-1$ and $b=-a$. Hence the asymptote is $y=-x-a$, or $x+y+a=0$. The result might have been obtained equally well by the substitution $x=my+b$.

---

* The equation $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0$ (2)

is obtained from $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0=0$ (3)

by changing x into $1/x$. The roots of (2) are the reciprocals of those of (3). Hence if a_0 and a_1 change and $\doteq 0$, two roots of (3) $\doteq 0$ and $\therefore$ two roots of (2) become infinite.

2. Find the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Ans. $y=\pm\frac{b}{a}x$.

57. Asymptotes by expansion. The following definition of an asymptote gives a better idea of the relation of the line to the curve:

Def. When the distance (measured parallel to an axis) between a line and a curve is infinitesimal as both recede to an infinite distance, the line is said to be an asymptote to the curve. Such lines may be rectilinear or curvilinear. If the equation of a curve, when y is expressed as a series of descending powers of x, take the form

$$y=ax+b+\frac{c}{x}+\frac{d}{x^2}+\ldots, \tag{1}$$

the line $y=ax+b$ will be a rectilinear asymptote. For the difference between the y of the curve and the y of the line is $c/x+d/x^2+\ldots$, which is infinitesimal when x is infinite.

The line $y=ax+b$ is also the limit of a tangent of the curve (1). For the slope of the tangent $=dy/dx=a-cx^{-2}-\ldots\doteq a$ for x infinite, and the y-intercept of the tangent $=y-x\,dy/dx=b+2cx^{-1}+\ldots\doteq b$.

The sign of the term c/x in (1) will determine whether the curve lies above or below the asymptote when x is very large.

If the equation take the form

$$y=ax^2+bx+c+\frac{d}{x}+\frac{e}{x^2}+\ldots,$$

there will be a curvilinear asymptote, viz., the parabola $y=ax^2+bx+c$.

Ex. 1. $y^3=x^3+3ax^2$, Fig. 38.

We have $y^3=x^3\left(1+\frac{3a}{x}\right)$, or $y=x\left(1+\frac{3a}{x}\right)^{\frac{1}{3}}$, which by the

binomial theorem (when $x|>|3a$)

$$=x\left(1+\frac{a}{x}-\frac{a^2}{x^2}\div\ldots\right), \quad \text{or} \quad y=x+a-\frac{a^2}{x}+\ldots;$$

$\therefore\ y=x+a$ is an asymptote. The curve lies below the asymptote when x is a large positive number, and above it when x is a large negative number.

2. The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Here $y=\pm\frac{bx}{a}\left(1-\frac{a^2}{x^2}\right)^{\frac{1}{2}}$

$$=\pm\frac{bx}{a}\left(1-\frac{a^2}{2x^2}-\ldots\right)=\pm\frac{bx}{a}\mp\frac{ab}{2x}\ldots$$

$\therefore$ the asymptotes are $y=\pm\frac{bx}{a}$.

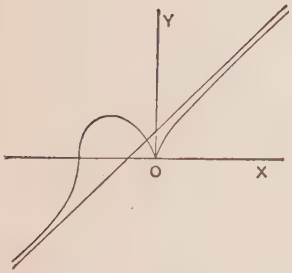

Fig. 38.

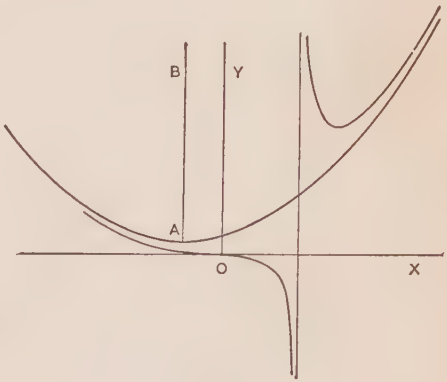

Fig. 39.

3. $4y(x-1)=x^3$, Fig. 39.

By division, $\quad 4y=\frac{x^3}{x-1}=x^2+x+1+\frac{1}{x-1}$.

When x is very large the last term is very small and $\doteq 0$, and the ordinate of the curve $\doteq$ that of the parabola $4y=x^2+x+1$, which is called a parabolic asymptote. (The line AB is the axis of the parabola.) It will be noticed that this curve is asymptotic to the given curve both when x is $+$ and when x is $-$. The line $x=1$ is a rectilinear asymptote, as will be seen from § 58.

58. Asymptotes parallel to the axes. Let the algebraical equation of a curve, freed (if necessary) from fractions and radicals affecting the coördinates, *and arranged in descending powers of* x, be

$$f_1(y)x^m+f_2(y)x^{m-1}+f_3(y)x^{m-2}+\ldots=0,$$

whence

$$f_1(y)+f_2(y)\frac{1}{x}+f_3(y)\frac{1}{x^2}+\ldots=0. \qquad (1)$$

If there is an asymptote parallel to the x-axis, y remains finite when x is infinite. Hence all the terms of (1) after the first become infinitesimal, and the y of the curve approaches a limit which satisfies $f_1(y)=0$, i.e., $y=$ the y of a line $y-a=0$, $y-a$ being a factor of $f_1(y)$. Hence, when the equation of a curve is arranged according to powers of x, the coefficient of the highest power, equated to zero, represents the asymptotes which are parallel to the x-axis. The asymptotes parallel to the y-axis may be found in the same way.

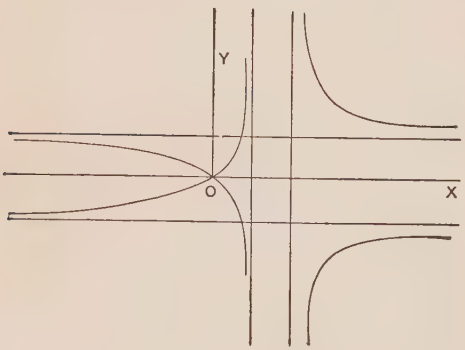

Fig. 40.

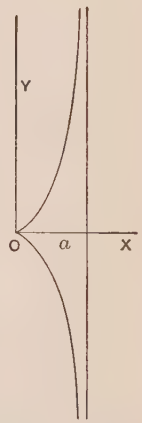

Fig. 41.

Ex. 1. $x^2y^2-3xy^2-x^2+2y^2=0$, Fig. 40.

Arranged according to powers of x the equation is

$$(y^2-1)x^2-3y^2x+2y^2=0.$$

and according to powers of y,

$$(x-1)(x-2)y^2-x^3=0.$$

Hence $y=\pm 1$, and $x=1$, $x=2$, are asymptotes parallel to the axes.

2 The cissoid $y^2(a-x)=x^3$, Fig. 41.

The line $a-x=0$, or $x=a$ is an asymptote parallel to the y-axis.

59. In any equation the terms of the highest degree, equated to zero, represent lines drawn through the origin. The equation which gives the slopes of these lines is found by substituting mx for y, or m for y/x. This is the same equation as that which determines the slopes of the asymptotes (§ 56). Hence the terms of the highest degree, equated to zero, represent lines drawn through the origin in the direction of the infinite branches of the curve.

EXAMPLES.

1. $x^3-y^3=3axy$. *Ans.* $y=x-a$.
2. $y^3=x^2y+2x^2$. $y=\pm x+1$, $y+2=0$.
3. $x^4=xy^3+3y^3$. $x-y=1$, $x+3=0$.
4. $x^2y=x^3+x+y$. $y=x$, $x=\pm 1$.
5. $x^4-y^4+x^2=4xy^2$. $x\pm y=1$.
6. $x^5=x^2y^3-(1-x)y^3$. $3(x-y)=1$, $2x=\pm\sqrt{5}-1$.
7. $axy=x^3-a^3$. $x=0$, $ay=x^2$.
8. $x^3+y^3=a^3$. $x+y=0$.
9. $x^3-27y^3=2x^2$. $3x-9y-2=0$.
10. $y+xy=x^3$. $x+1=0$, $y=x^2-x+1$.

11. $y=\tan x$. The y of the curve $=\infty$ when the $x=\frac{1}{2}\pi$, $\therefore$ the line $x=\frac{1}{2}\pi$ is an asymptote (§ 57). Similarly $x=(n+\frac{1}{2})\pi$, n any integer, is an asymptote. The same lines are asymptotes to $y=\sec x$.

12. Show that $y=1$ and $y=-1$ are asymptotes of $y=\tanh x$.

13. Show that the curve $2y=e^x$ is asymptotic to $y=\sinh x$, $y=\cosh x$, and $y=0$.

CHAPTER XIV.

TANGENT PLANES. TANGENTS TO CURVES IN SPACE.

60. Geometrical illustration of partial and total differentials. Let $z=f(x, y)$. Values of x and y determine z, and hence a point (x, y, z) in space referred to axes which we shall assume to be rectangular. Points thus obtained lie on a surface which is the locus of the equation $z=f(x, y)$. This surface is a geometrical representation of the function.

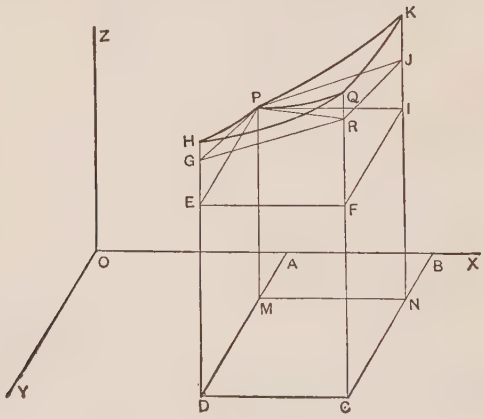

FIG. 42.

Let $OA=x$, $AM=y$, and $MP=z$. Then P is a point on the surface. Let $OB=x+dx$, $BC=y+dy$, and let the new value of z be CQ. Then Q is another point on the surface. The plane PF parallel to XOY cuts off $CF=MP$, hence FQ is Δz, the increment of z. Planes through P and Q parallel to XOZ and ZOY cut the surface in PH, HQ, QK, KP.

Draw PJ and PG tangents to PK, PH, and let $PGRJ$ be the plane through PJ, PG. If we suppose y to be constant, we are confined to the plane PN (produced if necessary); $PI=dx$ and PJ touches PK, hence IJ is d_xz. Similarly EG is d_yz. Let a line through the middle point of MC parallel to OZ meet the plane $PGRJ$. This line $=\frac{1}{2}(MP+CR)$ in the trapezium $PMCR$, and also $=\frac{1}{2}(DG+NJ)$ in the trapezium $GDNJ$. $\therefore FR=IJ+EG=d_xz+d_yz$. But $dz=d_xz+d_yz$ (§ 45). Hence FR is dz. The tangents of the angles IPJ, EPG are the partial derivatives of z with respect to x and y, i.e., they are $\partial z/\partial x$ and $\partial z/\partial y$.

61. Tangent plane. When Δz and dz are infinitesimal the latter is the part of the former, which contains the infinitesimals of the lowest order (§ 46). Thus FQ and FR correspond in the plane $PMCQ$ to Δy and dy of § 42. Hence the straight line PR touches the section of the surface made by the plane $PMCQ$, and therefore the plane $PGRJ$ is the locus of all such tangent lines at P, for dx and dy are any increments. Such a plane is defined to be the tangent plane at P.

Notice that if (x, y, z) is the point of contact, and dx, dy are any increments, $(x+dx, y+dy, z+dz)$ is any other point in the tangent plane.

62. Equation of the tangent plane. Let the equation of the surface be $f(x, y, z)=c$ or $u=c$. Differentiating,

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0.$$

Let (x_1, y_1, z_1) be the coördinates of the point of contact P, Fig. 42, (x, y, z) those of any other point in PR and therefore of any other point in the tangent plane. Then $x-x_1$, $y-y_1$, $z-z_1$ are proportional to dx, dy, dz.

$$\therefore \left(\frac{\partial u}{\partial x}\right)_1(x-x_1)+\left(\frac{\partial u}{\partial y}\right)_1(y-y_1)+\left(\frac{\partial u}{\partial z}\right)_1(z-z_1)=0 \quad (1)$$

is the equation of the tangent plane at (x_1, y_1, z_1).

Ex. To find the tangent plane at the point $(-1, 1, 2)$ on the surface $x^3-x^2y+y^2+z=1$.

$\partial u/\partial x=3x^2-2xy=5$ for the point $(-1, 1, 2)$,
$\partial u/\partial y=-x^2+2y=1$ for the point $(-1, 1, 2)$,
$\partial u/\partial z=1$.

Hence the tangent plane is

$$5(x+1)+(y-1)+(z-2)=0, \quad \text{or} \quad 5x+y+z+2=0.$$

63. Equations of the normal. The normal passes through (x_1, y_1, z_1) and is perpendicular to the tangent plane; hence its equations are

$$\frac{x-x_1}{\left(\frac{\partial u}{\partial x}\right)_1}=\frac{y-y_1}{\left(\frac{\partial u}{\partial y}\right)_1}=\frac{z-z_1}{\left(\frac{\partial u}{\partial z}\right)_1}. \tag{1}$$

64. Tangent plane at the origin. Conical points. Let the equation of the surface be freed (if necessary) from fractions and radicals affecting the coördinates. If the origin is on the surface the equation will contain no constant terms, and by substituting $r\cos\alpha$, $r\cos\beta$, $r\cos\gamma$ for x, y, z, it may be shown exactly as in § 52 that the terms of the first degree, equated to zero, represent the tangent plane at the origin. Similarly, if there are no terms of the first degree, those of the lowest degree present will represent a surface touching the given surface at the origin. This tangent surface is generally a cone, in which case the origin is called a conical point; but it may be two or more planes.*

As in § 54, it may be shown that the coördinates of a conical point or a point where there are two or more tangent planes will satisfy $\partial u/\partial x=0$, $\partial u/\partial y=0$, $\partial u/\partial z=0$, as well as the given equation $u=c$.

* A homogeneous equation with no constant term represents a locus of straight lines passing through the origin. For, if satisfied by x, y, z, it is satisfied by cx, cy, cz, the coördinates of any other point on the line joining the origin to (x, y, z).

Ex. 1. Of the surface $x^2-y^2-z^2+x^3=0$, $x^2-y^2-z^2=0$ is a tangent cone at the origin.

2. Of the surface $x^2-y^2-z^2-2yz=x^3$, $x+y+z=0$ and $x-y-z=0$ are tangent planes at the origin.

3. Find a conical point on the surface $x^3+y^2+2yz-3x-4z=2$.

Ans. (1, 2, −2).

65. Centre of a quadric. Exactly as in § 48 it may be shown that the centre of any surface whose equation $u=c$ is of the second degree is obtained by solving the simultaneous equations $\partial u/\partial x=0$, $\partial u/\partial y=0$, $\partial u/\partial z=0$.

Ex. Find the centre of $x^2-3y^2-z^2+4yz-4x+8y-6z=0$.

Ans. (2, 2, 1).

66. Curve in space. Let P and Q be two points near one another on a curve, P being (x, y, z) and Q $(x+\Delta x, y+\Delta y, z+\Delta z)$. Then $\Delta x=AB=CE=PG$, $\Delta y=ED=GF$, and $\Delta z=FQ$. Let the arc $PQ=\Delta s$ and the chord $PQ=q$. The tangent PT is the limit of position of the secant PQ when Q approaches coincidence with P. Let α, β, γ be the direction angles of PT. Then $\alpha=HPT$, and

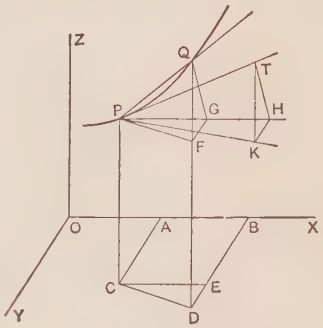

Fig. 43.

$$\cos\alpha=\text{Lt}\cos GPQ=\text{Lt}(\Delta x/q)$$
$$=\text{Lt}(\Delta x/\Delta s)\ (\S\ 17)\ =dx/ds.$$

Similarly $\cos\beta=dy/ds$, $\cos\gamma=dz/ds$.

Squaring and adding,

$$1=\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2, \quad \text{or} \quad ds^2=dx^2+dy^2+dz^2. \quad (1)$$

Draw TK parallel to ZO to meet the plane PFG, and KH parallel to YO. Then if dx is PH, dy is HK, dz is KT, and ds is PT.

67. If the coördinates of a point on a curve are given in terms of a fourth variable, dx, dy, and dz may be written down at once. Usually, however, a curve in space is given as the intersection of two surfaces. Let the surfaces be $u=c_1$ and $v=c_2$. Differentiating,

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0, \tag{1}$$

and

$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0. \tag{2}$$

If (x, y, z) is the point of contact P of a tangent plane, $(x+dx, y+dy, z+dz)$ is any other point Q in the plane. Hence if $P(x, y, z)$ is a point in the curve of intersection of the surfaces, and (1) and (2) are simultaneous in dx, dy, dz, $Q(x+dx, y+dy, z+dz)$ is any other point in the line of intersection of the tangent planes, and PQ is the tangent at P to the curve of intersection.

The plane which is perpendicular to the tangent line at the point of contact is called the normal plane of the curve.

Ex. Find the equations of the tangent to the curve

$$x^2-2y^2+yz+3=0, \quad xy-z^2+x+4=0$$

at the point $(1, -1, 2)$.

Differentiating the equations,

$$2x\,dx+(z-4y)dy+y\,dz=0,$$
$$(y+1)dx+x\,dy-2z\,dz=0,$$

or, for the point $(1, -1, 2)$,

$$2dx+6dy-dz=0, \quad dy-4dz=0,$$

whence

$$\frac{dx}{-23}=\frac{dy}{8}=\frac{dz}{2}.$$

Since dx, dy, dz are proportional to the direction cosines of the tangent, the equations of the tangent are

$$\frac{x-1}{-23}=\frac{y+1}{8}=\frac{z-2}{2}.$$

The normal plane at $(1, -1, 2)$ is

$$-23(x-1)+8(y+1)+2(z-2)=0, \quad \text{or} \quad 23x-8y-2z=27.$$

Examples.

1. Find the tangent plane at $(2, -1, 1)$ on the surface $x^2-2y^2+z=3$. *Ans.* $4x+4y+z=5$.

2. Find the tangent plane at $(-1, 1, 2)$ on $x^2+y^2-z^2-yz-4xy=0$. *Ans.* $6x-4y+5z=0$.

3. The tangent plane at any point of the central quadric $x^2/a^2+y^2/b^2+z^2/c^2=1$ is $x_1x/a^2+y_1y/b^2+z_1z/c^2=1$.

4. The tangent plane to the surface $xyz=a^3$ makes with the coördinate planes a tetrahedron of constant volume.

5. What are the tangent planes at the origin of the conoid $x^2-y^2=x^2z^2$? *Ans.* $x \pm y=0$.

6. Find the tangent planes to $x^4+y^4+z^4=3axyz$, (1) at $(0, 0, 0)$, (2) at (a, a, a).

Ans. (1) $x=0$, $y=0$, $z=0$, (2) $x+y+z=3a$.

Also to $x^3+y^3+z^3=3a^2x$ at the same points.

Ans. (1) $x=0$, (2) $y+z=2a$.

7. The sum of the squares of the intercepts of the tangent planes of the surface $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}=a^{\frac{2}{3}}$ on the axes is a^2.

8. $x=a \sin nz$, $y=a \cos nz$ are the equations of a helix (screw-thread) on a circular cylinder of radius a, the z-axis being the axis of the cylinder. Show that the equations of the tangent at any point are

$$\frac{x-x_1}{ny_1}=\frac{y-y_1}{-nx_1}=z-z_1,$$

and that the tangent makes a constant angle with the xy-plane.

9. Find the direction angles of the tangent to the curve of intersection of the surfaces of Ex. 6 at the point (a, a, a).

Ans. 0, 45°, 135°.

CHAPTER XV.

SUCCESSIVE DIFFERENTIATION.

68. Successive derivatives. The differential of $f(x)$ is (§ 27) $f'(x)\,dx$. Let $df'(x)=f''(x)\,dx$, $df''(x)=f'''(x)\,dx$, etc. The several functions $f'(x)$, $f''(x)$, $f'''(x)$, ... are called the first, second, third, ... derived functions, derivatives, or differential coefficients of $f(x)$.

Ex. 1. $f(x)=3x^3-2x+4$, $f'(x)=9x^2-2$, $f''(x)=18x$, $f'''(x)=18$, $f^{\text{iv}}(x)=0$.

2. $f(x)=\sin x$, $f'(x)=\cos x$, $f''(x)=-\sin x$, $f'''(x)=-\cos x$, etc.

3. $f(x)=\log(1+x)$, $f'(x)=1/(1+x)$, $f''(x)=-1/(1+x)^2$, etc.

4. $f(x)=e^x$, $f'(x)=e^x$, $f''(x)=e^x$, etc.

69. Successive differentials. Let y or $f(x)$ be a function of x, then $dy=f'(x)\,dx$. The differential of dy, or $d(dy)$, is written d^2y (read d-two y, or second dy); similarly, $d^3y\equiv d(d^2y)$.

Unless the variable x is given as a function of another variable it is assumed to be an independent variable—one to which arbitrary values may be assigned; dx is then an arbitrary increment of x. It is customary to take dx as of the same value in each successive differentiation, i.e., to treat the differential of the independent variable as a constant in differentiating. Hence, differentiating $dy=f'(x)dx$,

$d^2y = f''(x)\,dx\,.\,dx = f''(x)\,dx^2$, or $d^2y/dx^2 = f''(x)$.* (1)

Hence also, $d^3y = f'''(x)\,dx^3$, or $d^3y/dx^3 = f'''(x)$,

and similarly for the higher differentials. Thus the successive differential quotients are equal to the successive derivatives.

Ex. 1. $y = e^{ax}$, $dy = ae^{ax}\,dx$, $d^2y = a^2e^{ax}\,dx^2$, $d^3y = a^3e^{ax}\,dx^3$, . . ., $d^ny = a^ne^{ax}\,dx^n$.

2. $y = \cos x$, $dy/dx = -\sin x = \cos\left(x + \frac{\pi}{2}\right)$,

$$\therefore\ d^2y/dx^2 = \cos\left(x + 2\frac{\pi}{2}\right), \ldots, d^ny/dx^n = \cos\left(x + n\frac{\pi}{2}\right).$$

3. $y = x^n$, n a positive integer, $d^ny/dx^n = n!$

70. If x is not the independent variable—if it is itself a function of another variable—we cannot treat dx as a constant in successive differentiation. For if $x = f(\theta)$, $dx = f'(\theta)d\theta$, and is therefore a function of θ.

Differentiating both sides of $f'(x) = \dfrac{dy}{dx}$, we have

$$f''(x)\,dx = \frac{dx\,d^2y - dy\,d^2x}{dx^2}, \quad \text{by (D)},$$

$$\therefore\ f''(x) = \frac{dx\,d^2y - dy\,d^2x}{dx^3}. \tag{1}$$

Comparing with § 69 (1) we see that the d^2y/dx^2 obtained when x is the independent variable is equal to

$$\frac{dx\,d^2y - dy\,d^2x}{dx^3},$$

obtained when x is not the independent variable.

Ex. The cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. Considering y as a function $f(x)$ of x, to evaluate $f'(x)$ and $f''(x)$ when $\theta = \pi$.

$$dx = a(1 - \cos\theta)\,d\theta, \quad dy = a\sin\theta\,d\theta,$$

---

* Since dx is of arbitrary value it may be taken as infinitesimal, in which case d^2y is in general an infinitesimal of the same order as dx^2.

and, θ being independent variable,

$$d^2x = a \sin\theta\, d\theta^2, \quad d^2y = a\cos\theta\, d\theta^2.$$

$$\therefore f'(x) = \frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = 0 \quad \text{when} \quad \theta = \pi.$$

Substituting in (1), $f''(x) = -\dfrac{1}{a(1-\cos\theta)^2} = -\dfrac{1}{4a}$ when $\theta = \pi$.

EXAMPLES.

1. $y = ax^2 + bx + c$, $dy/dx = 2ax + b$, $d^2y/dx^2 = 2a$.
2. $y = (a+x)^3$, $dy/dx = 3(a+x)^2$, $d^2y/dx^2 = 6(a+x)$.
3. $y = x^2 \log x$, $d^3y/dx^3 = 2x^{-1}$.
4. $y = \cos ax$, $d^4y/dx^4 = a^4 \cos ax$.
5. $x = \sin^{-1}y$, $d^2x/dy^2 = y(1-y^2)^{-\frac{3}{2}}$, $d^2y/dx^2 = -y$.
6. If $f(x) = \sin x$, $f^{(n)}(x) = \sin\left(x + n\dfrac{\pi}{2}\right)$.
7. $f(x) = xe^x$, $f^{(n)}(x) = (x+n)e^x$.
8. $y = \log x$, $d^ny/dx^n = (-1)^{n-1}(n-1)!/x^n$.
9. If $y = a\cos nx + b \sin nx$, $d^2y/dx^2 + n^2y = 0$.
10. If $y = ae^{nx} + be^{-nx}$, $d^2y/dx^2 - n^2y = 0$.
11. If $y = e^{-x}\cos x$, $d^4y/dx^4 + 4y = 0$.
12. If $y = \sin^{-1}x$, $(1-x^2)d^2y/dx^2 - x\, dy/dx = 0$.
13. If $y = \tan^{-1}x$, $(1+x^2)d^2y/dx^2 + 2x\, dy/dx = 0$.
14. If $y = e^x \sin x$, $d^2y/dx^2 - 2dy/dx + 2y = 0$.
15. Given $x\, dy - y\, dx = r^2\, d\theta$, show that

$$x\, d^2y - y\, d^2x = 2r\, dr\, d\theta + r^2\, d^2\theta.$$

16. By differentiating

$$dx = \cos\phi\, ds, \quad \text{and} \quad dy = \sin\phi\, ds,$$

show that

$$(d^2x)^2 + (d^2y)^2 = (d\phi\, ds)^2 + (d^2s)^2.$$

17. $y^2 = 4ax$, $2y\, dy = 4a\, dx$, or $y\, dy = 2a\, dx$; differentiating again, $y\, d^2y + dy^2 = 0$ (dx being constant),

$$\therefore y\, d^2y + (2a\, dx/y)^2 = 0, \quad \text{or} \quad d^2y/dx^2 = -4a^2/y^3.$$

18. Given the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, show that

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.$$

d^2y/dx^2 may be found as in Ex. 17, or from dy/dx. Thus

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\frac{y - x\dfrac{dy}{dx}}{y^2}.$$

Substitute for dy/dx and reduce.

19. If $y = f(x)$ find $f''(x)$, given $x = a\cos\theta$, $y = b\sin\theta$.

Ans. $f''(x) = -b^4/a^2y^3$.

20. $a^2 + y^2 = 2xy$, $d^2y/dx^2 = a^2/(x-y)^3$.

21. $x^3 + y^3 = 3axy$, $d^2y/dx^2 = 2a^3xy/(ax - y^2)^3$.

CHAPTER XVI.

RATES.

71. Let y be a function of x of which Fig. 44 is the graph. When x increases by the amount Δx the change in y is Δy, and $\Delta y/\Delta x$ is called the average rate of change of y per unit of x (or briefly, the average x-rate of y) for the change Δx in x. When Δx is taken smaller and smaller and $\doteq 0$ the average rate $\Delta y/\Delta x$ is taken for a gradually diminishing change in x, and the limit of $\Delta y/\Delta x$, namely dy/dx, is defined to be the x-rate of y for the value x of the variable. Thus as x increases and reaches the value OA, the x-rate of the function y is dy/dx or $\tan \phi$. This is an instantaneous and variable rate, and is the same as the constant rate which y would have if P should henceforward move along the tangent PD.

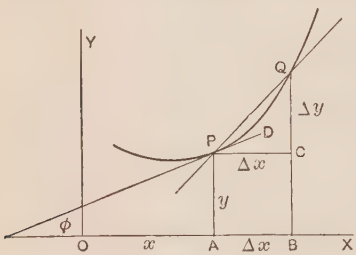

FIG. 44.

If $y=f(x)$, $dy/dx=f'(x)$; hence the x-rate of $f(x)$ is $f'(x)$, and for a similar reason the x-rate of any derivative is the succeeding derivative.

72. If y is a function of x, as x changes the function will increase or decrease according as its graph rises or falls, that is, according as dy is $+$ or $-$. Also dx is $+$ if x increases. Hence as x increases, y increases or decreases according as dy/dx is $+$ or $-$. Thus a $+$ value of the rate

implies that a function is increasing as its variable increases, and a $-$ value implies that the function is decreasing as the variable increases.

73. If x and y are functions of a third variable t,

$$dy/dx = (dy/dt)/(dx/dt).$$

Hence dy/dx is the quotient of the simultaneous rates of change of y and x, or dy and dx are proportional to the rates of y and x.*

If $y = f(x)$, and x is a function of t,

$$dy/dt = f'(x) \,.\, dx/dt,$$

which gives the rate of y in terms of that of x.

If $u = f(x, y)$, and x and y are functions of t, then, § 45 (5),

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt},$$

which gives the rate of u in terms of the rates of x and y.

74. If a point moving in a straight line is at a distance x from a fixed point in the line at the end of an interval of time whose measure is t, its velocity v is the t-rate of x, and is therefore dx/dt; and its acceleration a is the t-rate of v, and is therefore dv/dt. But

$$\frac{dv}{dt} = \frac{d\left(\frac{dx}{dt}\right)}{dt} = \frac{d^2x}{dt^2}. \quad \text{Also,} \quad \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v.$$

Hence $$v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx}.$$

Similarly the angular velocity and angular acceleration of a revolving body are $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ respectively.

Time rates are sometimes indicated by dots, $\dot{x}$ being the same as dx/dt, and $\ddot{x}$ the same as d^2x/dt^2.

* In some treatises dy and dx are defined to be rates.

Examples.

1. The ordinate of the curve $y=\sqrt{25-x^2}$ is moving parallel to the y-axis at the rate 2 in. per sec. At what rate is its length changing when $x=3$?

$\frac{dy}{dt}=-\frac{x}{\sqrt{25-x^2}}\frac{dx}{dt}=-\frac{3}{2}$ when $x=3$ and $dx/dt=2$. Hence y is decreasing at the rate of $1\frac{1}{2}$ in. per sec.

2. At what points on the curve $y=\log \sec x$ do x and y change at the same rate? *Ans.* $x=(n+\frac{1}{2})\pi$, n an integer.

3. Find the acceleration if (1) $v=u+bt$, (2) $x=ut+bt^2$, (3) $v^2=u^2+bx$, u and b being constants. *Ans.* (1) b, (2) $2b$, (3) $\frac{1}{2}b$.

4. If $x=a\cos(bt+c)$, show that the acceleration $=-b^2x$.

5. If $x=a\sinh(bt+c)$, show that the acceleration $=b^2x$.

6. Show that $\tan x$ always increases with x.

7. Three adjacent sides of a rectangular parallelepiped are 3, 4, 5 inches in length, and are each increasing at the rate of ·02 in. per in. per min. At what rate is the volume increasing?

Ans. 3·60 cu. in. per min.

8. One end of a ladder moves down a vertical wall with velocity v_1, while the other end moves along a horizontal plane with velocity v_2. Show that $v_1/v_2=\tan\theta$, where θ is the angle which the ladder makes with the vertical.

9. Two straight lines of railway intersect at an angle 60°. On one a train is 8 miles from the junction and moving towards it at the rate of 40 miles per hour, on the other a train is 12 miles from the junction and moving from it at the rate of 10 miles per hour. Is the distance of the trains from each other increasing or decreasing?

CHAPTER XVII.

MAXIMA AND MINIMA.

75. Suppose y to be a function of x and that x continually increases. Then (§ 72) y will increase or decrease according as dy/dx is $+$ or $-$. When dy/dx changes from $+$ to $-$, y ceases to increase and begins to decrease, and is then said to be a *maximum*; when dy/dx changes from $-$ to $+$, y ceases to decrease and begins to increase, and is then said to be a *minimum*. Now in order that a quantity may change sign it must become 0 or ∞ or $-\infty$;* hence as y becomes a max. or a min., dy/dx becomes 0 or ∞ or $-\infty$ and changes sign from $+$ to $-$ for a max. and from $-$ to $+$ for a min.

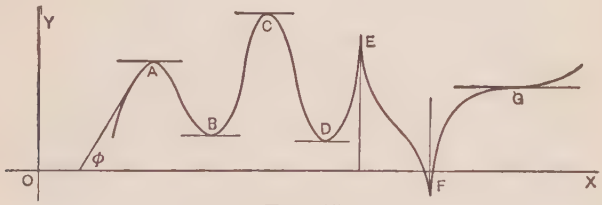

FIG. 45.

76. Suppose, for example, that the curve of Fig. 45 represents the graph of a function and that it is traced by a point moving from left to right so that dx is $+$. Then y decreases from A to B and dy/dx is $-$, between B and C y continually

* A quantity may change sign on account of finite discontinuity without passing through the value 0, but this occurs so rarely that we need not consider it further.

increases and dy/dx * is $+$; at B y ceases to decrease and begins to increase, dy/dx changes from $-$ to $+$ through the value 0, and y is a min. Similarly at C y is a max., and again a min. at D. At E dy/dx becomes ∞ and changes from $+$ to $-$, hence y is a max., and similarly y is a min. at F.

Points such as A, B, etc., are called turning points, and the max. and min. values of y are called turning values.

It will be noticed that a max. is not necessarily the greatest of all the values of y; it is greater than the values which immediately precede or follow it; and similarly a min. is not necessarily the least value of y.

77. To obtain the values of x which make a function y a max. or min. we must obtain dy/dx and find what values of x cause it to become zero or infinite. To distinguish the maxima from the minima we must determine whether dy/dx changes from $+$ to $-$ or from $-$ to $+$ as x passes through the critical value. In the former case y will be max., in the latter a min. It may happen, however, that dy/dx does not change sign, although it becomes 0 or ∞ (e.g., at G, Fig. 45), in which case y is neither a max. nor a min.

Ex. 1. $y=x^3-6x^2+9x+1$.

Here $dy/dx=3x^2-12x+9=3(x-1)(x-3)$.

When x is a little less than 1, $x-1$ is $-$ and $x-3$ is $-$, $\therefore$ dy/dx is $+$.

When $x=1$, dy/dx is 0. When x is a little more than 1, $x-1$ is $+$ and $x-3$ is $-$, $\therefore$ dy/dx is $-$. Hence dy/dx changes from $+$ to $-$ through 0 and $\therefore$ y is a max. when $x=1$. Substituting 1 for x in the given function we find the max. value of y to be 5. Similarly $x=3$ makes y a min., viz., 1.

2. $y=(x-1)^3$, $dy/dx=3(x-1)^2$; $\therefore$ $dy/dx=0$ when $x=1$, but does not change sign † when x passes through this value, $\therefore$ y is neither a max. nor a min.

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* It will be remembered that $dy/dx=\tan\phi$, and is therefore $+$ or $-$ according as ϕ is $+$ or $-$.

† $(x-a)^n$ changes sign with $x-a$ only when n is an odd integer, or a fraction whose numerator and denominator are both odd

3. $y=2+(x-1)^{\frac{2}{3}}$, $dy/dx=\dfrac{2}{3(x-1)^{\frac{1}{3}}}$; $\therefore$ dy/dx becomes $-\infty$ and changes from $-$ to $+$ as x passes through the value 1, hence $x=1$ makes y a min., viz., 2.

78. The sign of d^2y/dx^2 (=the x-rate of dy/dx) tells us at any time whether dy/dx is increasing or decreasing. If then the value of x which makes dy/dx equal to 0 also makes d^2y/dx^2 plus, we infer that dy/dx is increasing when it passes through 0, i.e., that dy/dx changes from $-$ to $+$, and hence that y is a min.; whereas, if the value of x which makes dy/dx equal to 0 also makes d^2y/dx^2 minus, we infer that dy/dx is decreasing when it passes through 0, i.e., that it changes from $+$ to $-$, and hence that y is a max.

Hence to distinguish the maxima from the minima we may find d^2y/dx^2, and in it substitute the values of x which make dy/dx equal to 0. Then for every $+$ result y is a min., and for every $-$ result y is a max.*

Ex. 1. In Ex. 1, § 77, $d^2y/dx^2=6x-12$, which is $-$ when $x=1$, and $+$ when $x=3$. Hence $x=1$ makes y a max. and $x=3$ makes y a min.

2. $y=x^3-7x^2+8x+30$, $dy/dx=3x^2-14x+8$. For a max. or a min. $3x^2-14x+8=0$, $\therefore$ $x=\frac{2}{3}$ or 4.

Also $d^2y/dx^2=6x-14$, which is $-$ when $x=\frac{2}{3}$ and $+$ when $x=4$; $\therefore$ $x=\frac{2}{3}$ makes y a max. and $x=4$ makes y a min.

79. It should be noticed:

(1) That max. and min. values must occur alternately in a continuous function, i.e., between two successive max. values there must be a min., and between two successive min. values there must be a max. Also of two values of x which make y a max. *or* a min., if one makes it a max. the other must make it a min.

(2) When y has a turning value, y^n (n a positive or negative integer) has a turning value. Thus a square-root sign

* If d^2y/dx^2 is 0 or ∞ it gives no information as to the turning values, and the test of § 77 must be applied.

affecting the whole of the variable part of a function may be disregarded in differentiating.

(3) A constant factor may be omitted from the function before differentiating, since it cannot affect the values of x for which the derivative is 0 or ∞.

Ex. $y=\pi x\sqrt{a^2-x^2}$. This $=\pi\sqrt{a^2x^2-x^4}$, and $\therefore$ y will be a max. or a min. when $a^2x^2-x^4$ is a max. or a min.; hence $2a^2x-4x^3=0$, $\therefore$ $x=0$, and $x=\pm a/\sqrt{2}$.

80. In the practical applications of this subject it will be necessary to form the function which is to have a turning value. It will frequently be obvious from the nature of the problem whether the result corresponds to a max. or a min.

Ex. 1. Of all arithmetical fractions, which one exceeds its square by the greatest quantity?

Let the fraction be x. Then $x-x^2$ is to be a max.

$\therefore$ $1-2x=0$, and hence $x=\frac{1}{2}$.

2. How to make with a given amount (area) of material a cylindrical box (with lid) which shall have the greatest possible volume.

We have the total surface of the cylinder given, call it s, and assume h for the height and x for the radius of the base.

Then $s=2\pi x^2+2\pi xh$, $\therefore$ $h=s/(2\pi x)-x$.

The volume $V=\pi x^2h=\frac{1}{2}sx-\pi x^3$.

$$\therefore\ dV/dx=\tfrac{1}{2}s-3\pi x^2=0, \text{ for a max.}$$

$$\therefore\ x=\sqrt{s/6\pi},\quad \text{whence}\quad h=2\sqrt{s/6\pi}.$$

Hence the height must = the diameter of the base and each $=2\sqrt{s/6\pi}$.

[Observe that in these examples the function which is to be a max. or a min. must be expressed in terms of some *one* variable with or without constants; in this case the function is πx^2h, where both x and h are variable, but there is a relation connecting x and h from which h may be obtained in terms of x; this when substituted in πx^2h gives a function with one variable.]

3. To find the greatest isosceles triangle that can be inscribed in a given circle.

Let ABC (Fig. 46) be an isosceles triangle inscribed in a circle of radius a and centre E. Let $DC=x$. Then

$$AD=\sqrt{AE^2-DE^2}=\sqrt{a^2-(x-a)^2}=\sqrt{2ax-x^2}.$$

$$\therefore \text{ area of } ABC=DC\,.\,AD=x\sqrt{2ax-x^2}=\sqrt{2ax^3-x^4}.$$

This will be a max. when $2ax^3-x^4$ is a max., § 79 (2), i.e., when $6ax^2-4x^3=0$, $\therefore\ x=\frac{3}{2}a$.

The triangle is easily shown to be equilateral.

4. One corner A of a rectangular piece of paper $ABCD$ (Fig. 47) is folded over to the side BC. Find when the crease EG is a min.

Let $AB=a$, $AE=x$, $EG=y$, $AGE=\theta$. Then $BEF=2\theta$.

$\therefore\ BE:EF=(a-x)/x=\cos 2\theta$ and $AE:EG=x/y=\sin\theta$.

Eliminating θ by the relation $\cos 2\theta=1-2\sin^2\theta$, we find

$$y^2=2x^3/(2x-a),$$

from which y is found to be a min. when $x=\frac{3}{4}a$.

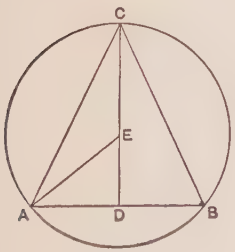

Fig. 46.

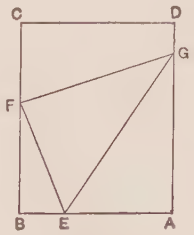

Fig. 47.

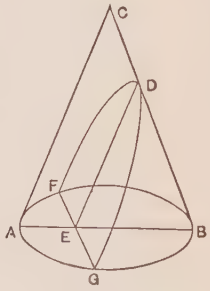

Fig. 48.

Similarly it may be shown that the area of the part folded over is a min. when $x=\frac{2}{3}a$.

5. To cut the parabola of greatest area from a given right circular cone, Fig. 48.

Let $AB=a$ and $EB=x$. The area $=\frac{2}{3}ED\,.\,FG$.

Now $EF^2=AE\,.\,EB=(a-x)x$, and ED is proportional to x.

$\therefore$ area varies as $x\sqrt{(a-x)x}$ or $\sqrt{ax^3-x^4}$, whence $x=\frac{3}{4}a$ for a max.

Examples.

1. x^2-3x+4, min. when $x=\frac{3}{2}$.
2. $x^5-5x^4+5x^3+1$, max. when $x=1$, min. when $x=3$.
3. $a+b(c-x)^{\frac{5}{3}}$, no turning value.
4. x^3-2x^2-4x+1, max., when $x=-\frac{2}{3}$, min. when $x=2$.
5. $(x-1)^3(x+2)^4$, max. when $x=-2$, min. when $x=-\frac{2}{7}$.
6. $(1+3x)/\sqrt{4+5x^2}$, max. when $x=\frac{1}{5}^2$.
7. $(x+2)^3/(x-3)^2$, min. when $x=13$.
8. $\sin\theta+\cos\theta$, max. when $\theta=\frac{1}{4}\pi$, min. when $\theta=\frac{5}{4}\pi$.
9. $\sin\theta/(1+\tan\theta)$, max. when $\theta=\frac{1}{4}\pi$, min. when $\theta=\frac{5}{4}\pi$.
10. $\sin\theta\sin(\alpha-\theta)$, max. when $\theta=\frac{1}{2}\alpha$.
11. $\sin^2\theta\cos^3\theta$, max. when $\sin\theta=\pm\sqrt{\frac{2}{5}}$, min. when $\theta=0$.
12. Min. value of $a\tan\theta+b\cot\theta=2\sqrt{ab}$.
13. Min. value of $a^2\sec^2\theta+b^2\operatorname{cosec}^2\theta=(a+b)^2$.
14. Min. value of $ae^{nx}+be^{-nx}=2\sqrt{ab}$.
15. Max. value of $\log x/x=1/e$.
16. What is the longest ordinate of the curve $a^2y^2=x^2(a^2-x^2)$, (Fig. 69)? *Ans.* $\frac{1}{2}a$.
17. Find the max. ordinates of the curves

$$(y-x)^2=x^3, \text{ Fig. 30, and } (y-x^2)^2=x^5, \text{ Fig. 31.}$$

Ans. $2^2/3^3$, $4^4/5^5$.

18. Find the max. ordinate of the curve

$$x^3+y^3=3axy, \text{ Fig. 28.}$$

Differentiating the equation and making $dy=0$ we have $x^2=ay$; from this and the equation of the curve we find the max. ordinate to be at the point $(a\sqrt[3]{2},\ a\sqrt[3]{4})$, the latter coördinate being the required value.

19. Find the max. ordinate of the curve $y^3=x^3+3ax^2$, Fig. 38. *Ans.* $\sqrt[3]{4}\,a$.

20. How could you cut out four equal squares from the corners of a given square so that the remaining area (the edges being turned up) would form a rectangular box of greatest volume?

Ans. Each side of the little squares $=\frac{1}{6}$ of a side of the given square.

21. Find the breadth and depth of the strongest beam that can be cut from a cylindrical log of diameter d, assuming that the

strength varies as the product of the breadth and the square of the depth.

Ans. Breadth $= \frac{1}{3}\sqrt{3}\,d$, depth $= \frac{1}{3}\sqrt{6}\,d$.

22. To cut out from a given sphere the cone of greatest volume.

Ans. Ht. of cone $= \frac{2}{3}$ diam. of sphere.

23. How could you cut a sector out of a circle so that the remainder of the circle would form the lateral surface of a cone of max. volume? *Ans.* Leave $\sqrt{\frac{2}{3}}$ of circumference.

24. What is the shortest distance of the line $y = x + 2$ from the parabola $y^2 = 4x$? *Ans.* $\frac{1}{2}\sqrt{2}$.

25. Assuming that the work of propelling a vessel in still water varies as the cube of the speed, what is the most economical rate of steaming against a current of speed v?

The expense for a given distance varies as x^3 and the time, and the latter varies inversely as $x - v$. *Ans.* $\frac{3}{2}v$.

CHAPTER XVIII.

CURVATURE.

81. Direction of curvature. Let it be supposed that the tangent of a curve rolls round the curve in such a way that the abscissa of the point of contact P continually increases. Let the tangent make an angle ϕ with the x-axis. Then a + value of d^2y/dx^2 (the x-rate of dy/dx) at P implies that dy/dx or $\tan\phi$, and therefore also ϕ, is increasing with x, or that the tangent is turning in the positive direction as x increases. In other words, the curve bends upward, or is concave upward, when d^2y/dx^2 is +, and bends downward, or is concave downward, when d^2y/dx^2 is −.

82. Point of inflexion. A point where a curve has ceased to bend upward and is about to bend downward, or *vice versa*, is called a point of inflexion. At such a point d^2y/dx^2 must change sign, and must therefore become $0, \infty$, or $-\infty$.*

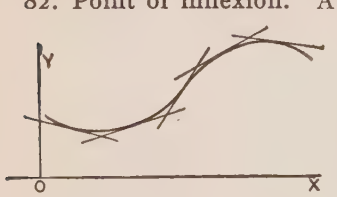

FIG. 49.

A tangent at a point of inflexion is sometimes called a stationary tangent, for, if the

* It is assumed in the above that x is the independent variable. If y is the independent variable, d^2x/dy^2 must change sign. If neither x nor y is independent, the quantity which must change sign is (§ 70)

$$(dx\,d^2y - dy\,d^2x)/dx^3.$$

tangent is supposed to roll round the curve, it comes to rest at such a point and reverses its motion.

83. If P, Fig. 50, is a point of inflexion, the secant through P and a point Q near P also passes through another point Q' near P. As the secant approaches the position of the tangent at P, Q and Q' approach coincidence with P at the same time. Hence the inflexional tangent is sometimes said to pass through three coincident points of the curve. A tangent at an ordinary point on a curve of the nth degree cannot meet the curve in more than $n-2$ other points; the tangent at a point of inflexion cannot meet the curve in more than $n-3$ other points, and in not more than $n-4$ other points if the point of contact is also a double point (as in Fig. 27).

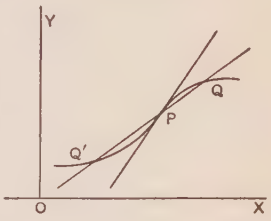

Fig. 50.

Ex. 1. $y=(x-1)^3$, $d^2y/dx^2=6(x-1)$. This is $-$ when $x<1$, 0 when $x=1$, $+$ when $x>1$; hence, as x increases, the curve bends downward until $x=1$, and upward afterwards; $\therefore$ there is a point of inflexion where $x=1$. Since y and dy/dx are also 0

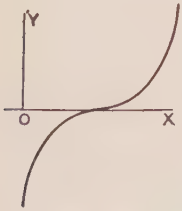

Fig. 51.

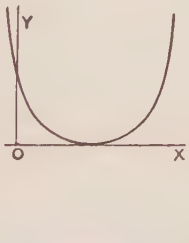

Fig. 52.

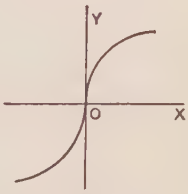

Fig. 53.

when $x=1$, the axis of x is the tangent at the point of inflexion (Fig. 51).

2. $y=(x-1)^4$, $d^2y/dx^2=12(x-1)^2$, which is 0 when $x=1$, but is never $-$, hence there is no point of inflexion (Fig. 52).

3. $y^3=x$, or $y=x^{\frac{1}{3}}$, $d^2y/dx^2=-\frac{2}{9}x^{-\frac{5}{3}}$, which becomes ∞ and changes from $+$ to $-$ when $x=0$, $\therefore$ the origin is a point of inflexion (Fig. 53).

4. $y=3x^4-4x^3-6x^2$, $d^2y/dx^2=12(3x^2-2x-1)$. Putting this $=0$ and solving for x, we get $x=-\frac{1}{3}$, $x=1$, which determine the points of inflexion.

Find the points of inflexion on the curves:

5. $a^2y=x(x^2-a^2)$, Fig. 17. *Ans.* (0, 0).
6. $xy=1+x^3$. $(-1, 0)$.
7. $(x+a)^2y=a^2x$. $(2a, \frac{2}{3}a)$.
8. $y=x(x-1)(x-2)$, Fig. 70. (1, 0).
9. $x^3-axy=a^3$. $(a, 0)$.
10. $(a^2+x^2)y=a^2x$. (0, 0), $(\pm a\sqrt{3}, \pm\frac{1}{4}a\sqrt{3})$.
11. $y^3=x^3+3ax^2$, Fig. 38. $(-3a, 0)$.
12. $x^3+y^3=a^3$. $(a, 0)$, $(0, a)$.
13. $x=y^3+3y^2$. $(2, -1)$.
14. $y^2=x^2(2x-1)$, Fig. 33. $(\frac{2}{3}, \pm\frac{2}{9}\sqrt{3})$.

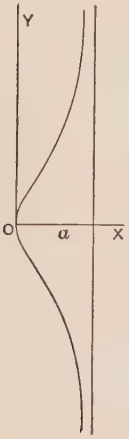

Fig. 54.

15. Show that at a point (x, y) a curve is convex or concave to the axis of x (i.e., with reference to the foot of the ordinate) according as $y\,d^2y/dx^2$ is $+$ or $-$.

16. Show that the curves $y=\sin x$, $y=\tan x$, meet the axis of x in points of inflexion.

17. Where are the points of inflexion of the curve $y=\cos x+\frac{1}{9}\cos 3x$?

Ans. Where $x=\frac{1}{4}\pi n$, n any integer not divisible by 4.

18. On the witch $y^2(a-x)=a^2x$ (Fig. 54), show that the points of inflexion are $(a/4, \pm a/\sqrt{3})$.

84. Centre, radius, and circle of curvature. Let P and Q be two points near one another on a curve $APQE$, Fig. 55, at which tangents and normals are drawn, the latter meeting in D. The limit of position C which D approaches as Q moves towards coincidence with P is called the *centre of curvature* of the curve at P, PC is

called the *radius of curvature*, and the circle with C as centre and PC as radius is called the *circle of curvature*.

The extremities of an infinitesimal arc are called consecutive points of the curve.* The normals at consecutive points are consecutive normals. Hence the centre of curvature is the limit of the point of intersection of consecutive normals.

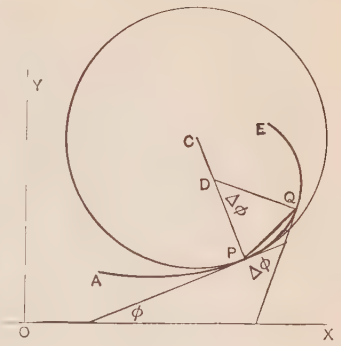

FIG. 55.

85. Let the length of PC be R. Let the tangents at P, Q make angles ϕ, $\phi+\Delta\phi$ with OX, then $\Delta\phi = PDQ$. Let s = the length of the arc of the curve measured from some point up to P, Δs = the arc PQ, and q = the chord PQ. Then $PD/q = \sin PQD/\sin \Delta\phi$. The limit of $\sin PQD = 1$, since the limit of PQD is a right angle. Hence the limit of $PD =$

$$\pounds(q/\sin \Delta\phi) = \pounds(\Delta s/\Delta\phi) \ (\S\ 17) \ = ds/d\phi.$$

$$\therefore \ R = ds/d\phi.$$

86. Imagine the tangent to be rolling round the curve, the point of contact having arrived at P. Then $d\phi/ds$ is the s-rate of ϕ, or the rate, in radians per unit length of the curve, at which the tangent is turning. This rate is taken as the measure of the curvature of the curve; hence $1/R$ measures the curvature at P. Since all normals of a circle intersect in the centre and are equal to the radius, the curvature of the circle of curvature is constant and $= 1/R$.

---

* The point consecutive to P is the point which is next considered and supposed subsequently to approach coincidence with P.

87. The circle of curvature generally crosses the curve at the point of contact, since in the circle the curvature is the same on both sides of the point of contact, which is not the case in the other curve except possibly at certain points, e.g., at the vertex of a conic section, where the circle of curvature does not cross the curve.

88. Length of the radius of curvature. We have seen that

$$R=\frac{ds}{d\phi} \quad (1); \qquad \text{we also have } \frac{dy}{dx}=\tan\phi. \tag{2}$$

Differentiating (2),

$$\frac{dx\,d^2y-dy\,d^2x}{dx^2}=\sec^2\phi\,d\phi=\left(\frac{ds}{dx}\right)^2 d\phi,$$

$$\therefore\ d\phi=\frac{dx\,d^2y-dy\,d^2x}{ds^2}, \tag{3}$$

$$\therefore\ R=\frac{ds^3}{dx\,d^2y-dy\,d^2x}. \tag{4}$$

We may generally take x as the independent variable and therefore make $d^2x=0$; also $ds^2=dx^2+dy^2$.

$$\therefore\ R=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\,d^2y}=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \tag{5}$$

The sign of R when found from (5) will be + or − according as d^2y/dx^2 is + or −, that is, according as the curve is concave upward or concave downward (§ 81).

If x and y are given in terms of a third variable m which is taken as independent, (4) may be expressed in the form

$$R=\frac{\left[\left(\frac{dx}{dm}\right)^2+\left(\frac{dy}{dm}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{dm}\frac{d^2y}{dm^2}-\frac{dy}{dm}\frac{d^2x}{dm^2}}. \tag{6}$$

Ex. 1. To find the radius of curvature at any point (x, y) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, Fig. 56.

By differentiating the equation of the ellipse we have

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y},\quad \frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}.$$

Substituting in (5), we have

$$R=-\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4},$$

which gives R in terms of x and y.* A more convenient expression may be found by substituting y^2 from the equation of the curve.

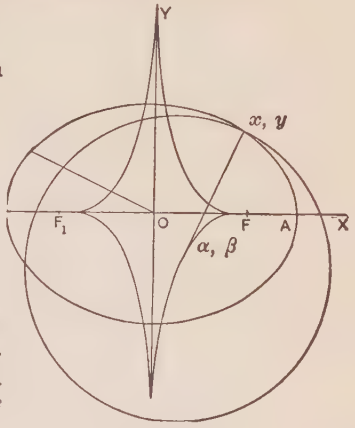

Fig. 56.

Then $$R=-\frac{(a^2-e^2x^2)^{\frac{3}{2}}}{ab},$$

where e is the eccentricity $\sqrt{a^2-b^2}/a$.

It is known that $(a^2-e^2x^2)^{\frac{1}{2}}$ = the semi-diameter parallel to the tangent, or perpendicular to the normal, at (x, y).

Calling this b_1 we have

$$R=-\frac{b_1^3}{ab}. \tag{7}$$

2. To find R at the origin of the curve $ay^3-3ax^2y=x^4$, Fig. 36, for the branch which touches the x-axis.

Let $y=mx$,† then $x=a(m^3-3m)$, $y=a(m^4-3m^2)$.

Thus x and y are known in terms of a third variable, and we require R from (6) for $m=0$. Differentiating,

$$dx/dm=a(3m^2-3)=-3a \quad \text{for} \quad m=0.$$
$$d^2x/dm^2=6am=0 \quad \text{for} \quad m=0.$$
$$dy/dm=a(4m^3-6m)=0 \quad \text{for} \quad m=0.$$
$$d^2y/dm^2=-6a.$$

* The sign of R will be $-$ or $+$ according as y is $+$ or $-$.

† See foot-note, p. 53.

Whence, from (6), $R=\frac{3}{2}a$.

Similarly $R=24a$ for the other branches of this curve at the origin ($m=\sqrt{3}$).

89. Coördinates of the centre of curvature. Let the coordinates be α and β. Then, Fig. 57,

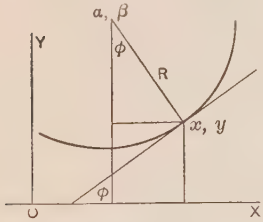

Fig. 57.

$$\alpha=x-R\sin\phi$$
$$=x-R\frac{dy}{ds}=x-\frac{dy\,ds^2}{dx\,d^2y-dy\,d^2x}. \quad (1)$$

$$\beta=y+R\cos\phi$$
$$=y+R\frac{dx}{ds}=y+\frac{dx\,ds^2}{dx\,d^2y-dy\,d^2x}. \quad (2)$$

Evolute—Involute. The locus of the centres of curvature of a curve is another curve which is called the evolute of the given curve, and the given curve is called the involute of the evolute.

Ex. To find the centre of curvature for any point (x, y) of an ellipse, and the equation of the evolute.

If x is the independent variable, (1) and (2) become

$$\alpha=x-\frac{dy}{dx}\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}, \quad \beta=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}},$$

which for the ellipse give

$$\alpha=\left(\frac{a^2-b^2}{a^4}\right)x^3, \quad \beta=-\left(\frac{a^2-b^2}{b^4}\right)y^3.$$

Solving for x and y and substituting in the equation of the ellipse we obtain

$$(a\alpha)^{\frac{2}{3}}+(b\beta)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$$

for the equation of the evolute (see Fig. 56). If x and y are substituted for α and β, the equation becomes

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}.$$

90. Properties of the evolute. (A) *Every normal of a curve touches the evolute at the centre of curvature.*

$$\alpha = x - R\sin\phi, \quad \therefore\ d\alpha = dx - R\cos\phi\, d\phi - \sin\phi\, dR.$$

But $$dx = ds\cos\phi = R\, d\phi\cos\phi.$$

$$\therefore\ d\alpha = -\sin\phi\, dR. \qquad (1)$$

Similarly $$d\beta = \cos\phi\, dR. \qquad (2)$$

$$\therefore\ d\beta/d\alpha = -\cot\phi = -dx/dy.$$

Hence the tangent of the evolute at (α, β) has the same slope as the normal at (x, y) on the involute, and (α, β) is on both lines, therefore they coincide.

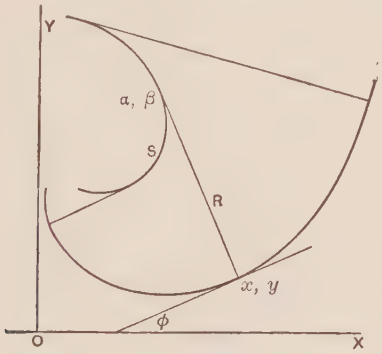

Fig. 58.

(B) *As long as the radii of curvature of a curve continue to increase or to decrease, the difference of any two is equal to the arc of the evolute included between them.*

Let the arc of the evolute be S. Then

$$dS = \sqrt{d\alpha^2 + d\beta^2} = \pm dR,$$

from (1) and (2). Suppose R to be increasing. Then $dS = dR$, hence S and R can only differ by a constant, and therefore any increment of S is equal to the corresponding increment of R. Similarly if R is decreasing, the increment of S is equal to the decrement of R.

From these properties of the evolute it will be obvious that if one of the tangents of the evolute were supposed to roll round the curve, a tracing-point in it would describe the involute. Thus although a given curve can have only one evolute, it can have any number of involutes. The involute might also be described by a tracing-point in a string which is kept stretched at the same time that it is unwound ("evolved") from the evolute.

Ex. The radii of curvature at the extremities of the axes of an ellipse are, § 88 (7), a^2/b and b^2/a. Hence the whole length of the evolute is

$$4\left(\frac{a^2}{b}-\frac{b^2}{a}\right)=4\left(\frac{a^3-b^3}{ab}\right).$$

EXAMPLES.

1. At any point of the parabola $y^2=4ax$ show that $R=-2\sqrt{r^3/a}$, where r is the focal distance $(=a+x)$ of the point. Hence it may

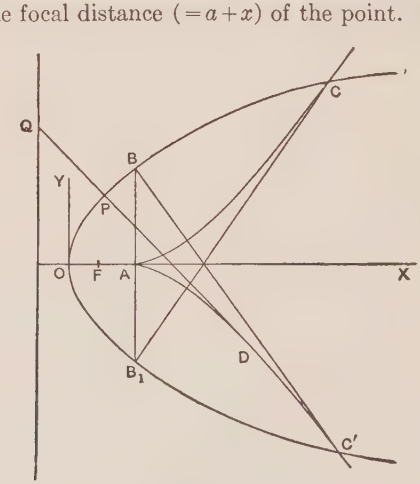

FIG. 59.

be shown that $R=$ twice the intercept on the normal between the directrix and the curve.

2. Prove that for $y^2=4ax$, $\alpha=2a+3x$, $\beta=-y^3/4a^2$.

3. Show that the evolute of the parabola is the semi-cubical parabola $27ay^2=4(x-2a)^3$, Fig. 59.

4. Show that C, C' are $(8a, \pm 4\sqrt{2}a)$, and that they are the centres of curvature of B', B.

5. Show that the arc $AC=2a(3\sqrt{3}-1)$.

6. Show that $R=(1+a^2)^{\frac{3}{2}}/2b$ *at the origin* on the curve

$$y=ax+bx^2+cx^3+\ldots$$

or

$$x=ay+by^2+cy^3+\ldots$$

7. Find R at the origin of the following curves:

(1) The parabola $y^2=4ax$, or $x^2=4ay$. *Ans.* $2a$.

(2) $y^2=x^2(1+2x)$, Fig. 32. We have

$$y=x(1+2x)^{\frac{1}{2}}=x(1+x-\tfrac{1}{2}x^2+\ldots). \qquad \pm\sqrt{2}$$

(3) $y^2=x^4(1+2x)$, Fig. 34. $\pm\frac{1}{2}$.

(4) $(y-x^2)^2=x^5$, Fig. 31. $\frac{1}{2}$.

(5) $(y-x)^2=x^3$, Fig. 30. 0.

8. Find the R of $x=y^2+y^3$ when $x=2$. *Ans.* $-\frac{1}{4}\sqrt[3]{26}$.

9. Find R at the point of maximum ordinate on the curve $y^3=x^3+3ax^2$, Fig. 38. *Ans.* $-\sqrt[3]{2}a$.

10. Show that $R=2\sqrt{2}a$ on the branch of the curve $ay^4-ax^2y^2=x^5$, Fig. 37, which touches $y=x$ at the origin.

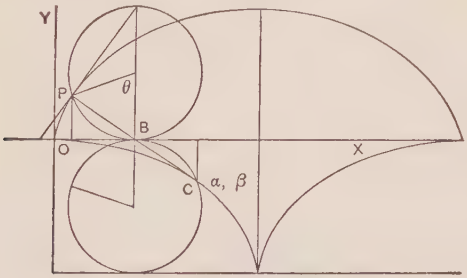

FIG. 60

11. The equation $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$ represents a common parabola, the origin being a point on the directrix, and the axes tangents to the curve. Show that $R=2(ax+by)^{\frac{3}{2}}/ab$.

[Fractional indices may be avoided by using $x=a\cos^4\theta$, $y=b\sin^4\theta$.]

12. Find R at any point of the curve $x = a\cos\theta$, $y = a\sin\theta$.

13. In the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (Fig. 18) show that $R = 3\sqrt[3]{axy}$ = 3 times the perpendicular from the origin on the tangent (Ex. 3, p. 46).

14. Show that the radius of curvature at any point of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is $-4a\sin\frac{1}{2}\theta$ = twice the normal PB (Fig. 60).

15. Also that $\alpha = a(\theta + \sin\theta)$, $\beta = -a(1 - \cos\theta)$, and hence (see Fig. 20) that the evolute is an equal cycloid.

16. Show that $R = \infty$ at a point of inflexion.

17. At any point of a curve $R = \dfrac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2 - (d^2s)^2}}$. (See Ex. 16, p. 70.)

CHAPTER XIX.

INTEGRATION. ELEMENTARY ILLUSTRATIONS.

91. Prop. The limit, when n is infinite, of the sum of n infinitesimals of the same sign is not changed if the infinitesimals are replaced by equivalent ones (§ 15).

Let the given infinitesimals be $\alpha_1, \alpha_2, \ldots$ and let $\beta_1, \beta_2, \ldots$ be equivalents. By a theorem of algebra $(\beta_1+\beta_2+\ldots)/(\alpha_1+\alpha_2+\ldots)$ lies in value between the greatest and the least of the fractions* $\beta_1/\alpha_1, \beta_2/\alpha_2, \ldots$ But each of these fractions $\doteq 1$ by hypothesis. Hence

$$\pounds(\beta_1+\beta_2+\ldots)=\pounds(\alpha_1+\alpha_2+\ldots),$$

or

$$\pounds\Sigma\beta=\pounds\Sigma\alpha.$$

Hence (§ 16) the limit of the sum depends only upon the infinitesimals of the lowest order.

92. In particular, if y is a function, and Δy and dy the infinitesimal increment and differential corresponding to the infinitesimal increment of the variable, then (§ 42) $\Delta y=dy+I$, and hence $\pounds\Sigma\,\Delta y=\pounds\Sigma\,dy$. This will be further considered in the following article, and a special notation will be employed for the limit of a sum.

---

* Let β_n/α_n be the greatest of the fractions, and let it $= r$. Also suppose the α's to be all positive. Then

$$\beta_1<r\alpha_1, \quad \beta_2<r\alpha_2, \quad \ldots, \quad \beta_n=r\alpha_n, \ldots$$

Hence, the symbol Σ indicating the sum of all terms of a single type,

$$\Sigma\beta<r\Sigma\alpha, \quad \text{or} \quad \Sigma\beta/\Sigma\alpha<r.$$

Similarly $\Sigma\beta/\Sigma\alpha>$ the least of the fractions.

93. Let $F(x)$ be a function of x, $f(x)$ its derivative, and suppose $F(x)$ and $f(x)$ to be continuous from $x=a$ to $x=b$. When x changes from a to b the change in $F(x)$ is $F(b)-F(a)$, or, in symbols,

$$\Big[F(x)\Big]_a^b = F(b)-F(a).$$

Suppose that x changes by the successive addition of infinitesimal increments. When x has the increment dx the corresponding increment of $F(x)$ is, § 42 (2), $f(x)\,dx+I$, where I stands for the higher infinitesimals. Hence $F(b)-F(a)=$ the limit of the sum of all such terms as $f(x)\,dx$, while x changes from a to b, dx approaching its limit 0 and the number of terms being infinite. Let this sum-limit be expressed by $\int_a^b f(x)\,dx$. Then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b)-F(a).$$

Hence, $f(x)$ being a function of x which is continuous from $x=a$ to $x=b$, to find the limit of the sum of all such terms as $f(x)\,dx$ when x changes from a to b we must seek the function $F(x)$ of which the differential is $f(x)\,dx$, substitute therein b and a successively for x, and subtract the second result from the first.

This process, which is analogous to summation,* is called integration (the making of a *whole* from infinitesimal parts); $F(x)$ is called the integral of $f(x)\,dx$, and $f(x)\,dx$ is called an element of the integral; a and b are called the limits † of the integration; $\int_a^b f(x)\,dx$ is read "integral from a to b (or between a and b) of $f(x)\,dx$."

---

* Historically, the symbol $\int$ is the old form of the letter *s*, the initial letter of the word *sum*.

† This meaning of the word limit is not the same as that employed elsewhere. It here signifies a value of the variable at one end of its range.

It should be noticed that dx is here regarded as an infinitesimal increment of x, and that the element or differential $f(x)\,dx$ is (§ 42) the increment of the function $F(x)$

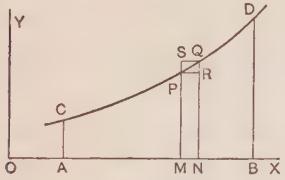

Fig. 61.

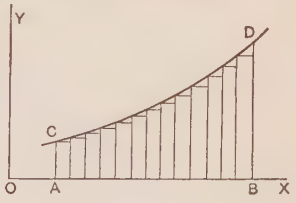

Fig. 62.

when the higher infinitesimals are omitted or disregarded. The practical applications of integration depend upon the fact that the element can be written down when $F(x)$ is unknown.

Illustrations.

94. Areas of curves. Let $y=f(x)$ be the equation of a continuous curve CD. Let $OA=a$, $OB=b$, $OM=x$, $MP=y$, $MN=dx$, $RQ=\Delta y$, and let it be required to find the area $ABDC$.

When x has the increment dx, the increment of the area is $MNQP=$ rectangle $MR+PRQ$. $MR=y\,dx$, and $PRQ<SR$ which $=dx\,\Delta y$ and is therefore a higher infinitesimal. Hence the element of the area is $y\,dx$.*

$$\therefore \text{ the area } ABDC=\int_a^b y\,dx=\int_a^b f(x)\,dx=\Big[F(x)\Big]_a^b,$$

where $F(x)$ is the function of which the differential is $f(x)\,dx$; e.g., the function of which the differential is $x^n dx$ is $\dfrac{1}{n+1}x^{n+1}$ except when $n=-1$, in which case it is $\log x$.

* Observe that the omission of the higher infinitesimals is equivalent to supposing y to remain constant while x increases by dx. More generally, the element of $dx(P_1+i_1)(P_2+i_2)\ldots$ is $dx\,.\,P_1P_2\ldots$, and $P_1, P_2\ldots$ may, in obtaining the element, be regarded as constant while x changes to $x+dx$.

In other words, we imagine the area to be divided into narrow strips by lines drawn parallel to OY, express the area of a strip as a differential or infinitesimal element (all infinitesimals of an order higher than the first being omitted) and then integrate. The whole area is seen to be the limit of the sum of the rectangles as their breadth $\doteq 0$ and their number becomes infinite.

Ex. 1. To find the area OBD of the curve $y=x^3$, Fig. 63, the limits of x being 0 and 1.

The area $= \int_0^1 y\,dx = \int_0^1 x^3\,dx = \left[\tfrac{1}{4}x^4\right]_0^1 = \tfrac{1}{4}$, i.e., the area is one-fourth of the square on OB.

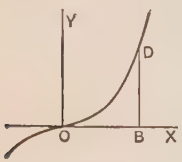

Fig. 63.

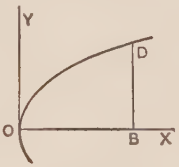

Fig. 64.

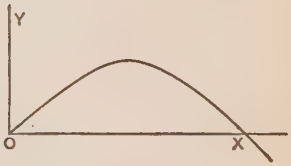

Fig. 65.

2. The area of the parabola $y^2=4ax$, Fig. 64; from $x=0$ to $x=h$ is

$$\int_0^h y\,dx = \int_0^h \sqrt{4a}\,.\,x^{\frac{1}{2}}\,dx = \left[\sqrt{4a}\,.\,\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^h$$

$$= \tfrac{2}{3}\sqrt{4a}\,.\,h^{\frac{3}{2}} = \tfrac{2}{3}h\,.\sqrt{4ah} = \tfrac{2}{3}OB\,.\,BD$$

= two-thirds of the rectangle having the same base and height.

3. The area of the curve $y=\sin x$, Fig. 65, from $x=0$ to $x=\pi$ is

$$\int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi = 2,$$

i.e., twice the square on the maximum ordinate.

95. Volumes of solids of revolution. Suppose the curve, Fig. 61, to revolve about OX and generate a solid. The rectangles MR, MQ generate cylinders of infinitesimal thickness dx, and radii y, $y+\Delta y$, and therefore of volume $\pi y^2 dx$.

$\pi(y+\Delta y)^2\,dx$. The latter $=\pi y^2\,dx$ when the higher infinitesimals are omitted. Hence the volume element $=\pi y^2\,dx$.

$$\therefore \text{ the whole volume} = \pi\int_a^b y^2\,dx.$$

In other words, we imagine the solid to be divided into thin slices by planes perpendicular to OX, express the volume of a slice as a differential or infinitesimal element, and then integrate. The whole volume is thus the limit of the sum of cylinders of volume $\pi y^2\,dx$, i.e., of the cylinders formed by the revolution of the rectangles of Fig. 62.

Ex. 1. The volume formed by the revolution of OBD, Fig. 63, round OX is

$$\pi\int_0^1 y^2dx = \pi\int_0^1 x^6dx = \pi\Big[\tfrac{1}{7}x^7\Big]_0^1 = \tfrac{1}{7}\pi.$$

2. When the area of the parabola $y^2=4ax$ from $x=0$ to $x=h$ revolves about OX the volume is

$$\pi\int_0^h y^2dx = \pi\int_0^h 4ax\,dx = \pi\Big[4a\,.\,\tfrac{1}{2}x^2\Big]_0^h = \tfrac{1}{2}\pi(4ah)h = \tfrac{1}{2}\pi BD^2\,.\,OB,$$

i.e., one-half of the cylinder having the same base and height.

CHAPTER XX.

FUNDAMENTAL INTEGRALS. 1.

96. We have seen that

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b,$$

$F(x)$ being the function of which $f(x)\,dx$ is the differential. If the limits are not expressed,* we may write

$$\int f(x)\,dx = F(x),$$

and hence $\int$ may be regarded as a symbol which indicates the operation of going from the differential $f(x)\,dx$ back to the primitive function $F(x)$, or of finding the antidifferential of $f(x)\,dx$, or the antiderivative of $f(x)$. By this operation we can discover only the variable part of the primitive function; e.g., $\int 2x\,dx = x^2$, or x^2+1, or x^2+c, where c may be any constant (any quantity independent of x). To every integral thus obtained from a differential there should $\therefore$ be added a constant, the value of which must depend upon special *data;* we should then write

$$\int f(x)\,dx = F(x)+c.$$

* The integral is said to be *definite* when the limits are expressed, *indefinite* when they are not expressed.

If we are hereafter to substitute limits, the constant c need not be expressed, inasmuch as it would disappear in subtracting. We shall accordingly, as a rule, omit the constant, but its presence is always understood.

97. The various processes by which integrals are obtained consist almost entirely in so changing the form of given differentials as to make them appear as particular cases of the fundamental ones given below.

Differentials.	**Integrals.**	
$d(v^n) = nv^{n-1}\,dv$	(A), $\therefore \int v^n\,dv = \frac{v^{n+1}}{n+1}$, $n \neq -1$	(a)
$d\sqrt{v} = \frac{dv}{2\sqrt{v}}$	(B), $\therefore \int \frac{dv}{2\sqrt{v}} = \sqrt{v}$	(b)
$d\left(\frac{1}{v}\right) = -\frac{dv}{v^2}$	(B_1), $\therefore \int \frac{dv}{v^2} = -\frac{1}{v}$	(b_1)
$d(a^v) = A\,a^v\,dv$	(E), $\therefore \int a^v\,dv = a^v/A$, $A = \log_e a$	(e)
$d(e^v) = e^v\,dv$	(F), $\therefore \int e^v\,dv = e^v$	(f)
$d(\log v) = \frac{dv}{v}$	(G), $\therefore \int \frac{dv}{v} = \log v$	(g)
$d(\sin v) = \cos v\,dv$	(H), $\therefore \int \cos v\,dv = \sin v$	(h)
$d(\cos v) = -\sin v\,dv$	(I), $\therefore \int \sin v\,dv = -\cos v$	(i)
$d(\tan v) = \sec^2 v\,dv$	(J), $\therefore \int \sec^2 v\,dv = \tan v$	(j)
$d(\cot v) = -\operatorname{cosec}^2 v\,dv$	(K), $\therefore \int \operatorname{cosec}^2 v\,dv = -\cot v$	(k)

Differentials.	Integrals.
$d(\sec v) = \sec v \tan v\, dv$ (L),	$\therefore \int \sec v \tan v\, dv = \sec v$ (l)
$d(\operatorname{cosec} v) = -\operatorname{cosec} v \cot v\, dv$ (M),	$\therefore \int \operatorname{cosec} v \cot v\, dv = -\operatorname{cosec} v$ (m)
$d\left(\sin^{-1}\frac{v}{a}\right) = \frac{dv}{\sqrt{a^2-v^2}}$ (N),	$\therefore \int \frac{dv}{\sqrt{a^2-v^2}} = \sin^{-1}\frac{v}{a}$ (n)
$d\left(\tan^{-1}\frac{v}{a}\right) = \frac{a\,dv}{a^2+v^2}$ (P),	$\therefore \int \frac{dv}{a^2+v^2} = \frac{1}{a}\tan^{-1}\frac{v}{a}$ (p)
$d\left(\sec^{-1}\frac{v}{a}\right) = \frac{a\,dv}{v\sqrt{v^2-a^2}}$ (Q),	$\therefore \int \frac{dv}{v\sqrt{v^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{v}{a}$ (q)

To these may be added: *

$$\int \frac{dv}{\sqrt{v^2 \pm a^2}} = \log\left(v + \sqrt{v^2 \pm a^2}\right), \qquad (r)$$

$$\left.\begin{aligned} \int \frac{dv}{a^2-v^2} &= \frac{1}{2a}\log\left(\frac{a+v}{a-v}\right), \quad v| < |a, \\ &= \frac{1}{2a}\log\left(\frac{a+v}{v-a}\right), \quad v| > |a, \end{aligned}\right\} \qquad (s)$$

and

$$\int \frac{dv}{v\sqrt{a^2 \pm v^2}} = \frac{1}{a}\log\left(\frac{v}{a+\sqrt{a^2 \pm v^2}}\right). \qquad (t)$$

We add the hyperbolic equivalents of (r), (s), and (t).

$$\left.\begin{aligned} \int \frac{dv}{\sqrt{v^2+a^2}} &= \sinh^{-1}\frac{v}{a}, \\ \int \frac{dv}{\sqrt{v^2-a^2}} &= \cosh^{-1}\frac{v}{a}, \end{aligned}\right\} \qquad (r')$$

* Formulæ (r), (s), (t) should be committed to memory with the others, as they are of fundamental importance. It will be seen later that they may be deduced from the preceding formulæ. Compare carefully (n) and (r), (p) and (s), (q) and (t). Notice that $\int \frac{dv}{v^2-a^2} = -\int \frac{dv}{a^2-v^2}$ and is therefore known from (s).

$$\left.\begin{aligned}\int\frac{dv}{a^2-v^2} &= \frac{1}{a}\tanh^{-1}\frac{v}{a}, \quad v|<|a,\\ \text{and}\qquad &= \frac{1}{a}\coth^{-1}\frac{v}{a}, \quad v|>|a,\end{aligned}\right\} \tag{s'}$$

$$\left.\begin{aligned}\int\frac{dv}{v\sqrt{a^2+v^2}} &= -\frac{1}{a}\sinh^{-1}\frac{a}{v} = -\frac{1}{a}\operatorname{cosech}^{-1}\frac{v}{a},\\ \int\frac{dv}{v\sqrt{a^2-v^2}} &= -\frac{1}{a}\cosh^{-1}\frac{a}{v} = -\frac{1}{a}\operatorname{sech}^{-1}\frac{v}{a}.\end{aligned}\right\} \tag{t'}$$

CHAPTER XXI.

FUNDAMENTAL INTEGRALS. II.

EXAMPLES. FORMULÆ (*a*) TO (*g*).

1. $\int ax^3dx = \frac{1}{4}ax^4$, $\int (ax^3+b)\,dx = \frac{1}{4}ax^4+bx$,

$$\int \frac{2dx}{x^3} = \int 2x^{-3}\,dx = \frac{2x^{-2}}{-2} = -\frac{1}{x^2}, \quad \int \frac{dx}{x^2} = -\frac{1}{x}.$$

2. $\int (x^2-a^2)^2dx = \int (x^4-2a^2x^2+a^4)\,dx = \frac{1}{5}x^5 - \frac{2}{3}a^2x^3 + a^4x$.

3. $\int (x^2-2)^{\frac{3}{2}}x\,dx = \frac{1}{2}\int (x^2-2)^{\frac{3}{2}}\,d(x^2-2) = \frac{1}{5}(x^2-2)^{\frac{5}{2}}$.

4. $\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\int \frac{d(a^2-x^2)}{2\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$, by (*b*).

5. $\int \frac{x\,dx}{a^2-x^2} = -\frac{1}{2}\int \frac{d(a^2-x^2)}{a^2-x^2} = -\frac{1}{2}\log(a^2-x^2)$, by (*g*).

6. $\int \frac{(1-x^2)^2}{x}dx = \int \frac{1-2x^2+x^4}{x}dx = \int \left(\frac{1}{x}-2x+x^3\right)dx$

$$= \log x - x^2 + \tfrac{1}{4}x^4.$$

7. $\int e^{-2x}\,dx = -\frac{1}{2}\int e^{-2x}\,d(-2x) = -\frac{1}{2}e^{-2x}$, by (*f*).

8. $\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^{-x^2}\,d(-x^2) = -\frac{1}{2}e^{-x^2}$.

9. $\int ax^{\frac{5}{2}}\,dx = \frac{2}{7}ax^{\frac{7}{2}}$.

10. $\int x^{-\frac{4}{3}}\,dx = -3x^{-\frac{1}{3}}$.

11. $\int \frac{x\,dx}{a^2+x^2} = \log \sqrt{a^2+x^2}$.

12. $\int (a+bx)^3\,dx = \frac{(a+bx)^4}{4b}$.

13. $\int \sqrt{2ax-x^2}\,(a-x)\,dx = \frac{1}{3}(2ax-x^2)^{\frac{3}{2}}$.

14. $\int e^{ax}\,dx = e^{ax}/a$.

15. $\int \frac{dx}{a-x} = \log\left(\frac{1}{a-x}\right)$.

16. $\int \frac{dx}{(a-x)^4} = \frac{1}{3(a-x)^3}$.

17. $\int \frac{x-2}{x\sqrt{x}}dx = \frac{2}{\sqrt{x}}(x+2)$.

18. $\int \frac{x^2\,dx}{a^3-x^3} = \frac{1}{3}\log\left(\frac{1}{a^3-x^3}\right)$.

19. $\int (\log x)^n \frac{dx}{x} = \frac{(\log x)^{n+1}}{n+1}$.

20. $\int \frac{dx}{x \log x} = \log(\log x)$.

21. $\int_0^a a\,dx = a^2$.

22. $\int_1^4 3\sqrt{x}\,dx = 14$.

23. $\int_0^1 x^{-\frac{1}{2}}\,dx = 2$.

24. $\int_1^\infty \frac{dx}{x^4} = \frac{1}{3}$.*

25. $\int_a^{ae} \frac{dx}{x} = 1$.

26. $\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$.

27. $\int_2^a (ax+1)dx = \frac{1}{2}a^3 - a - 2$.

28. $\int_0^1 e^x\,dx = e-1$.

29. $\int_0^1 x^n\,dx = \frac{1}{n+1}$.

30. $\int_a^{2a} \sqrt{x-a}\,dx = \frac{2}{3}a^{\frac{3}{2}}$.

31. $\int_{-a}^a (a \pm x)^n\,dx = \frac{(2a)^{n+1}}{n+1}$.

32. $\int_0^2 \frac{x\,dx}{4+x^2} = \frac{1}{2}\log 2$.

33. $\int_a^\infty \frac{dx}{(a+x)^n} = \frac{1}{(n-1)(2a)^{n-1}}, \quad n>1$.

34. $\int_0^1 (a+bx+cx^2)x\,dx = \frac{1}{12}(6a+4b+3c)$.

---

* It is implied in § 92 that a and b are assigned values of x. In this example $\frac{1}{3}$ is to be understood as the limit of $\int_1^b \frac{dx}{x^4}$ when b is infinite.

35. $\int_0^{\frac{1}{2}a} \frac{x\,dx}{(a^2-x^2)^2} = \frac{1}{6a^2}$.

36. $\int_0^a \frac{x\,dx}{\sqrt{a^2+x^2}} = a(\sqrt{2}-1)$.

37. $\int_1^2 \frac{\log x}{x} dx = \frac{1}{2}(\log 2)^2$.

38. $\int_0^\infty x^3\, e^{-x^4} dx = \frac{1}{4}$.

39. $\int_0^a (a-x)^2\, x^{\frac{1}{2}}\, dx = \frac{16}{105} a^{\frac{7}{2}}$.

40. $\int_0^{2a} \frac{dx}{\sqrt{2a-x}} = 2\sqrt{2a}$.

41. $\int_0^a \frac{(a-x)dx}{\sqrt{2ax-x^2}} = a$.

42. $\int_0^1 3^{3x} dx = 26/\log 27$

CHAPTER XXII.

FUNDAMENTAL INTEGRALS. III.

Examples. Formulæ (h) to (m).

1. $\int \sin 3\theta \, d\theta = \frac{1}{3}\int \sin 3\theta \, d(3\theta) = -\frac{1}{3}\cos 3\theta.$

2. $\int \cos 5\theta \cos 3\theta \, d\theta = \frac{1}{2}\int (\cos 8\theta + \cos 2\theta) \, d\theta$

$$= \tfrac{1}{2}\left(\frac{\sin 8\theta}{8} + \frac{\sin 2\theta}{2}\right).$$

3.* $\int \frac{d\theta}{\sin\theta\cos\theta} = \int \frac{\sec^2\theta \, d\theta}{\tan\theta} = \int \frac{d\tan\theta}{\tan\theta} = \log(\tan\theta).$

4. $\int \frac{d\theta}{\sin\theta} = \int \frac{d\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \int \frac{d(\frac{1}{2}\theta)}{\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \log(\tan\tfrac{1}{2}\theta),$ by Ex. 3.

5. $\int \frac{d\theta}{\cos\theta} = \int \frac{d(\frac{1}{2}\pi+\theta)}{\sin(\frac{1}{2}\pi+\theta)} = \log\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta),$ by Ex. 4,

or $= \log(\sec\theta + \tan\theta).$†

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* Integrals 3–11 deserve special attention on account of their frequent occurrence.

† This important integral may also be treated as follows:

$$\int \sec\theta \, d\theta = \int \frac{\sec^2\theta \, d\theta}{\sqrt{\tan^2\theta+1}} = \log(\sec\theta+\tan\theta), \text{ by } (r).$$

The integrals of Exs. 5, 4, 3 may also be expressed thus (see footnote, p. 36):

$$\int \frac{d\theta}{\cos\theta} = \lambda(\theta), \quad \int \frac{d\theta}{\sin\theta} = \lambda(\theta - \tfrac{1}{2}\pi), \quad \int \frac{d\theta}{\sin\theta\cos\theta} = \lambda(2\theta - \tfrac{1}{2}\pi).$$

Numerical values of $\lambda(\theta)$ are given at the end of the book.

6. $\int \tan\theta\, d\theta = \int \frac{\sin\theta\, d\theta}{\cos\theta} = -\log\cos\theta = \log\sec\theta.$

7. $\int \cot\theta\, d\theta = \int \frac{\cos\theta\, d\theta}{\sin\theta} = \log\sin\theta.$

8. $\int \sin^2\theta\, d\theta = \frac{1}{2}\int (1-\cos 2\theta)d\theta = \frac{1}{2}(\theta - \frac{1}{2}\sin 2\theta).$

9. $\int \cos^2\theta\, d\theta = \frac{1}{2}\int (1+\cos 2\theta)d\theta = \frac{1}{2}(\theta + \frac{1}{2}\sin 2\theta).$

10. $\int \tan^2\theta\, d\theta = \int (\sec^2\theta - 1)d\theta = \tan\theta - \theta.$

11. $\int \sin^2\theta\cos^2\theta\, d\theta = \frac{1}{4}\int \sin^2 2\theta\, d\theta = \frac{1}{8}\int (1-\cos 4\theta)d\theta$
$= \frac{1}{8}(\theta - \frac{1}{4}\sin 4\theta).$

12. $\int \frac{\cos\theta\, d\theta}{\sin^5\theta} = \int (\sin\theta)^{-5} d(\sin\theta) = -\frac{1}{4\sin^4\theta}.$

13. $\int \cos(3\theta - 1)d\theta = \frac{1}{3}\sin(3\theta - 1).$ 14. $\int \sec^2 4\theta\, d\theta = \frac{1}{4}\tan 4\theta.$

15. $\int \frac{\sin^3\theta}{\cos^5\theta} d\theta = \frac{1}{4}\tan^4\theta.$ 16. $\int \cos^2 n\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}(\sin 2n\theta)/n.$

17. $\int \frac{d\theta}{\sin\theta + \cos\theta} = \frac{1}{2}\sqrt{2}\log\tan\,(\frac{1}{8}\pi + \frac{1}{2}\theta) = \frac{1}{2}\sqrt{2}\,\lambda(\theta - \frac{1}{4}\pi).$

18. $\int \sqrt{1+\cos\theta}\, d\theta = 2\sqrt{2}\sin\tfrac{1}{2}\theta.$

19. $\int \frac{d\theta}{\sqrt{1+\cos\theta}} = \sqrt{2}\log\tan\tfrac{1}{4}(\pi + \theta) = \sqrt{2}\,\lambda(\tfrac{1}{2}\theta).$

20. $\int \sqrt{1 \pm \sin\theta}\, d\theta = 2(\sin\tfrac{1}{2}\theta \mp \cos\tfrac{1}{2}\theta).$

21. $\int \frac{d\theta}{1+\cos\theta} = \tan\tfrac{1}{2}\theta.$ 22. $\int \frac{d\theta}{1+\sin\theta} = \tan(\tfrac{1}{2}\theta - \tfrac{1}{4}\pi).$

23. $\int \sin 5\theta \cos 3\theta \, d\theta = -\frac{1}{4}(\cos 2\theta + \frac{1}{4}\cos 8\theta)$.

24. $\int \sin 3\theta \sin 2\theta \, d\theta = \frac{1}{2}(\sin\theta - \frac{1}{5}\sin 5\theta)$.

25. $\int \sin^5\theta \, d\theta = \int (1-\cos^2\theta)^2 \sin\theta \, d\theta = -\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta$.

26.* $\int \sin^4\theta \, d\theta = \int \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$

$= \frac{1}{4}\int [1 - 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)] = \frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta$.

27. $\int \tan^3\theta \, d\theta = \int (\sec^2\theta - 1)\tan\theta \, d\theta = \frac{1}{2}\tan^2\theta + \log\cos\theta$.

28. $\int \frac{\sin^3\theta \, d\theta}{\cos^2\theta} = -\int \frac{(1-\cos^2\theta)\, d\cos\theta}{\cos^2\theta} = \sec\theta + \cos\theta$.

29. $\int \cos^4\theta \sin^3\theta \, d\theta = -\int \cos^4\theta(1-\cos^2\theta)\, d\cos\theta = -\frac{1}{5}\cos^5\theta + \frac{1}{7}\cos^7\theta$.

30. $\int \frac{d\theta}{\sin^2\theta\cos^2\theta} = \tan\theta - \cot\theta$. 31. $\int \frac{\sin\frac{1}{2}\theta}{\sin\theta}\, d\theta = \log\tan\frac{1}{4}(\pi+\theta) = \lambda(\frac{1}{2}\theta)$.

32. $\int \frac{\sin^2\theta \, d\theta}{\cos\theta} = \log\tan(\frac{1}{4}\pi + \frac{1}{2}\theta) - \sin\theta = \lambda(\theta) - \sin\theta$.

33. $\int \frac{d\theta}{\cos^4\theta} = \int (1+\tan^2\theta)\sec^2\theta \, d\theta = \tan\theta + \frac{1}{3}\tan^3\theta$.

34. $\int_0^{\pi} \sin\theta \, d\theta = 2$. 35. $\int_0^{\pi} \cos\theta \, d\theta = 0$.

36. $\int_0^{\frac{1}{4}\pi} \frac{d\theta}{\cos^2\theta} = 1$. 37. $\int_0^{\frac{1}{4}\pi} \tan^2\theta \, d\theta = 1 - \frac{1}{4}\pi$.

38. $\int_0^{\frac{1}{2}\pi} \sin^2\theta \, d\theta = \frac{1}{4}\pi = \int_0^{\frac{1}{2}\pi} \cos^2\theta \, d\theta$. 39. $\int_0^{\frac{1}{4}\pi} \frac{\sin\theta \, d\theta}{\cos^2\theta} = \sqrt{2} - 1$.

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* Compare the methods in Exs. 25 and 26 according as the index is odd or even.

40. $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{d\theta}{\tan\theta} = \frac{1}{2}\log 2 = \cdot 347$*. 41. $\int_{0}^{\frac{1}{3}\pi} \sec^3\theta \tan\theta \, d\theta = 2\frac{1}{3}$.

42. $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^3\theta \, d\theta = \frac{1}{2}(1-\log 2) = \cdot 522$.

43. $\int_{0}^{\frac{1}{4}\pi} \tan^4\theta \, d\theta = \cdot 119$.

44. $\int_{0}^{\frac{1}{3}\pi} \frac{d\theta}{\cos 2\theta} = \frac{1}{2}\log\tan {}^{5}_{12}\pi = \cdot 658 = \frac{1}{2}\lambda(\frac{1}{3}\pi)$.

45. $\int_{0}^{\frac{1}{2}} \sec\theta \, d\theta = \cdot 521$.

46. $\int_{\frac{1}{3}\pi}^{\pi} \frac{(1+\cos\theta)\, d\theta}{\theta+\sin\theta} = \log\left(\frac{2\pi}{\pi+2}\right) = \cdot 201$.

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* Use the tables at the end of the book.

CHAPTER XXIII.

FUNDAMENTAL INTEGRALS. IV.

EXAMPLES. FORMULÆ (n) TO (t').

1. $$\int\frac{dx}{a^2-x^2}=\int\frac{1}{2a}\left[\frac{1}{a+x}+\frac{1}{a-x}\right]dx=\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right).$$

It also $$=\int\frac{1}{2a}\left[\frac{1}{a+x}-\frac{1}{x-a}\right]dx=\frac{1}{2a}\log\left(\frac{a+x}{x-a}\right).$$

This is (s).

2. $$\int\frac{dx}{\sqrt{4-3x^2}}=\frac{1}{\sqrt{3}}\int\frac{d(x\sqrt{3})}{\sqrt{2^2-(x\sqrt{3})^2}}=\frac{1}{\sqrt{3}}\sin^{-1}\left(x\frac{\sqrt{3}}{2}\right).$$

3. $$\int\frac{x\,dx}{\sqrt{1-x^4}}=\tfrac{1}{2}\int\frac{d(x^2)}{\sqrt{1-(x^2)^2}}=\tfrac{1}{2}\sin^{-1}(x^2).$$

4. $$\int\frac{dx}{\sqrt{2ax-x^2}}=\int\frac{d(x-a)}{\sqrt{a^2-(x-a)^2}}=\sin^{-1}\left(\frac{x-a}{a}\right).$$

5. $$\int\frac{dx}{\sqrt{x^2\pm2ax}}=\int\frac{d(x\pm a)}{\sqrt{(x\pm a)^2-a^2}}=\log\left(x\pm a+\sqrt{x^2\pm2ax}\right).$$

6. $$\int\frac{dx}{2ax-x^2}=\frac{1}{2a}\log\left(\frac{x}{2a-x}\right).$$ 7. $$\int\frac{dx}{x^2+2ax}=\frac{1}{2a}\log\left(\frac{x}{x+2a}\right).$$

8. $$\int\frac{dx}{x\sqrt{x^2-a^2}}=\int\frac{dx}{x^2\sqrt{1-\frac{a^2}{x^2}}}=-\frac{1}{a}\int\frac{d\left(\frac{a}{x}\right)}{\sqrt{1-\left(\frac{a}{x}\right)^2}}=-\frac{1}{a}\sin^{-1}\left(\frac{a}{x}\right).$$ *

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*The integral is also $\frac{1}{a}\sec^{-1}\frac{x}{a}$. These apparently different results differ only by a constant (in this case $\pi/2a$), and therefore have the same differential

In a similar manner deduce (t) from (r) and (t') from (r').

9. $\int \frac{x^2\,dx}{1+x^6} = \frac{1}{3}\tan^{-1} x^3.$

10. $\int \frac{x^2\,dx}{1-x^6} = \frac{1}{6}\log\left(\frac{1+x^3}{1-x^3}\right).$

11. $\int \frac{x\,dx}{\sqrt{a^4-x^4}} = \frac{1}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right).$

12. $\int \frac{dx}{e^x+e^{-x}} = \tan^{-1} e^x.$

13. $\int \frac{dx}{\sqrt{e^{2x}-1}} = \sec^{-1} e^x.$

14. $\int \frac{dx}{\sqrt{1-e^{2x}}} = -\operatorname{sech}^{-1} e^x.$

15. $\int \frac{dx}{2x^2-2x+1} = \frac{1}{2}\int \frac{dx}{(x^2-x+\frac{1}{4})+\frac{1}{4}} = \frac{1}{2}\int \frac{d(x-\frac{1}{2})}{(x-\frac{1}{2})^2+(\frac{1}{2})^2}$
$= \tan^{-1}(2x-1).$

16. $\int \frac{dx}{\sqrt{1-x-x^2}} = \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right).$

17. $\int \frac{dx}{\sqrt{1+x+x^2}} = \sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$

18. $\int \frac{dx\,\sqrt{x^2-a^2}}{x} = \sqrt{x^2-a^2} - a\sec^{-1}\frac{x}{a}.$

[Rationalize the numerator.]

19. $\int \frac{dx\,\sqrt{a^2-x^2}}{x} = \sqrt{a^2-x^2} - a\operatorname{sech}^{-1}\frac{x}{a}.$

20. $\int dx\sqrt{\frac{a-x}{a+x}} = \sqrt{a^2-x^2} + a\sin^{-1}\frac{x}{a}.$

21. $\int \frac{dx}{x}\sqrt{\frac{x+a}{x-a}} = \sec^{-1}\frac{x}{a} + \cosh^{-1}\frac{x}{a}.$

22. $\int \frac{dx}{x\sqrt{4x^2-9}} = \frac{1}{3}\sec^{-1}(\tfrac{2}{3}x).$

23. $\int \frac{dx}{\sqrt{5x^4-3x^2}} = \frac{1}{\sqrt{3}}\sec^{-1}\left(x\sqrt{\frac{5}{3}}\right).$

24. $\int \frac{x^{\frac{1}{2}}\,dx}{\sqrt{5-4x^3}} = \frac{1}{3}\int \frac{d(2x^{\frac{3}{2}})}{\sqrt{5-(2x^{\frac{3}{2}})^2}} = \frac{1}{3}\sin^{-1}\left(\frac{2x^{\frac{3}{2}}}{\sqrt{5}}\right).$

25. $\int_0^2 \frac{3\,dx}{4+9x^2} = \frac{1}{2}\tan^{-1} 3 = \cdot 624.$ 26. $\int_0^1 \frac{x\,dx}{\sqrt{1-x^4}} = \frac{1}{4}\pi = \cdot 785.$

27. $\int_0^1 \frac{dx}{2+5x^2} = \frac{1}{10}\sqrt{10}\tan^{-1}(\frac{1}{2}\sqrt{10}) = \cdot 318.$

28. $\int_0^{\frac{1}{2}} \frac{x\,dx}{1-x^4} = \frac{1}{4}\log\frac{5}{3} = \cdot 128.$ 29. $\int_1^2 \frac{dx}{x^2+4x} = \frac{1}{4}\log\frac{5}{3} = \cdot 128.$

30. $\int_1^2 \frac{x\,dx}{x^4+4} = \frac{1}{4}\tan^{-1}\frac{3}{4} = \cdot 161.$ 31. $\int_1^4 \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\frac{3}{4} = \cdot 322.$

32. $\int_1^\infty \frac{dx}{x\sqrt{2x^2-1}} = \frac{1}{4}\pi = \cdot 785.$ 33. $\int_0^1 \frac{x^{\frac{1}{2}}\,dx}{\sqrt{8-4x^3}} = \frac{1}{12}\pi = \cdot 262.$

CHAPTER XXIV.

INTEGRATION OF RATIONAL FRACTIONS.

98. An algebraic fraction is rational when it contains no surd expressions involving the variable.

If the fraction is improper, it must first be reduced to a mixed quantity.

Ex. 1. $\dfrac{x^4}{1+x^2}=x^2-1+\dfrac{1}{1+x^2}$. $\therefore \int\dfrac{x^4\,dx}{1+x^2}=\frac{1}{3}x^3-x+\tan^{-1}x$.

2. $\int\dfrac{x^5\,dx}{1+x^2}=\int\left(x^3-x+\dfrac{x}{1+x^2}\right)dx=\frac{1}{4}x^4-\frac{1}{2}x^2+\frac{1}{2}\log(1+x^2)$.

When the fraction is a proper fraction, it should, in general, be decomposed into partial fractions. See Appendix, Note A.

EXAMPLES.

1. $\dfrac{x^2+3x+1}{x(x-1)(x+2)}=-\dfrac{1}{2}\cdot\dfrac{1}{x}+\dfrac{5}{3}\cdot\dfrac{1}{x-1}-\dfrac{1}{6}\cdot\dfrac{1}{x+2}$.

$$\therefore \int\frac{x^2+3x+1}{x(x-1)(x+2)}dx=-\tfrac{1}{2}\log x+\tfrac{5}{3}\log(x-1)-\tfrac{1}{6}\log(x+2)$$
$$=\tfrac{1}{3}\log\frac{(x-1)^5}{\sqrt{x^4+2x^3}}.$$

2. $\int\dfrac{(1+3x)\,dx}{x+2x^2+x^3}=\int\left(\dfrac{1}{x}-\dfrac{1}{1+x}+\dfrac{2}{(1+x)^2}\right)dx$

$$=\log x-\log(1+x)-\frac{2}{1+x}=\log\left(\frac{x}{1+x}\right)-\frac{2}{1+x}.$$

3. $\int \frac{dx}{(x-a)(x-b)} = \int \frac{dx}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right) = \frac{1}{a-b}\log\left(\frac{x-a}{x-b}\right)$.

4. $\int \frac{dx}{1+3x+2x^2} = \log\left(\frac{1+2x}{1+x}\right)$.

5. $\int \frac{8x^5\,dx}{1+2x^2} = x^4 - x^2 + \frac{1}{2}\log(1+2x^2)$.

6. $\int \frac{(2x+1)\,dx}{x(x+1)(x+2)} = \log(x+1)\sqrt{\frac{x}{(x+2)^3}}$.

7. $\int \frac{dx}{x(1+x)^2} = \log\left(\frac{x}{1+x}\right) + \frac{1}{1+x}$.

8. $\int \frac{dx}{x^4(1+x^2)} = \frac{1}{x} - \frac{1}{3x^3} + \tan^{-1}x$.

9. $\int \frac{(3x^2-1)\,dx}{x^2-3x+2} = 3x + 11\log(x-2) - 2\log(x-1)$.

10. $\int \frac{x^2\,dx}{x^3+5x^2+8x+4} = \log(1+x) + \frac{4}{2+x}$.

11. $\int \frac{(x^3+1)\,dx}{x(x^3-1)} = \frac{2}{3}\log(x^3-1) - \log x$.

12. $\int \frac{(1+x)\,dx}{x(1+x^2)} = \tan^{-1}x + \log\frac{x}{\sqrt{1+x^2}}$.

13. $\int \frac{2x\,dx}{1+x+x^2+x^3} = \log\frac{\sqrt{1+x^2}}{1+x} + \tan^{-1}x$.

14. $\int \frac{2x\,dx}{(1+x^2)(3+x^2)} = \log\left(\frac{1+x^2}{3+x^2}\right)^{\frac{1}{2}}$.

15. $\int \frac{x\,dx}{(x^2+3x+2)^2} = \frac{3x+4}{x^2+3x+2} + 3\log\left(\frac{x+1}{x+2}\right)$.

CHAPTER XXV.

INTEGRATION BY SUBSTITUTION.

99. To assist in bringing certain differentials under forms already considered various substitutions are employed, the most important of which will be mentioned in this chapter.

Ex. 1. $\int \frac{dx}{ax+bx^n}$. Let $x=\frac{1}{z}$, then $dx=-\frac{dz}{z^2}$, and $\frac{dx}{x}=-\frac{dz}{z}$.

Substituting, we have

$$-\int \frac{z^{n-2}\,dz}{az^{n-1}+b}=\frac{\log\,(az^{n-1}+b)}{-a(n-1)}, \quad \text{by } (g).$$

$$=\frac{1}{a(n-1)}\log\left(\frac{x^{n-1}}{a+bx^{n-1}}\right), \quad \text{when } z \text{ is replaced by } \frac{1}{x}.$$

2. Making the same substitution and integrating by (a) we have

$$\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}}=\frac{x}{a^2(a^2-x^2)^{\frac{1}{2}}}, \quad \int \frac{dx}{(x^2\pm a^2)^{\frac{3}{2}}}=\pm\frac{x}{a^2(x^2\pm a^2)^{\frac{1}{2}}}.$$

3. $\int \frac{dv}{\sqrt{v^2\pm a^2}}$. Let $\sqrt{v^2\pm a^2}=z$. Then $v^2\pm a^2=z^2$ and $2v\,dv=2z\,dz$.

$$\therefore \frac{dv}{z}=\frac{dz}{v}=\frac{d(v+z)}{v+z}. \quad \therefore \int\frac{dv}{z}=\log\,(v+z),$$

or

$$\int \frac{dv}{\sqrt{v^2\pm a^2}}=\log\,(v+\sqrt{v^2\pm a^2}).$$

Thus formula (r) is deduced from (g).

100. Binomial differentials. Any expression of the form $(ax^p+bx^q)^r\,dx$, the indices being positive or negative, integers or fractions, may be called a binomial differential. For convenience in making the following statements it will be best to suppose the binomial differential to be given in the form

$$x^m(ax^n+b)^r\,dx.$$

This can be integrated immediately in the following cases:

(1) When r is a positive integer, expand by the Binomial Theorem.

(2) When $\dfrac{m+1}{n}$ is a positive integer, let $ax^n+b=z$.

(3) When $\dfrac{m+1}{n}+r$ is a negative integer, let $ax^n+b=x^nz$.

101. When the differential is a function of $a+bx$ let $a+bx=z^n$, where n is the L.C.M. of the denominators of the indices.

Ex. $$\int\frac{dx}{(1+x)^{\frac{2}{3}}+(1+x)^{\frac{1}{2}}}=\int\frac{2z\,dz}{z^3+z}, \quad \text{if} \quad 1+x=z^2,$$

$$=2\int\frac{dz}{1+z^2}=2\tan^{-1}z=2\tan^{-1}\sqrt{1+x}.$$

102. In $\dfrac{dx}{(Ax^2+B)\sqrt{ax^2+b}}$ let $ax^2+b=x^2z^2$.

103. $\left.\begin{matrix}\sin^m\theta\,d\theta,\\ \sin^m\theta\cos^n\theta\,d\theta,\end{matrix}\right\}$ m odd and $+$, let $\cos\theta=z$, $\therefore\ -\sin\theta\,d\theta=dz$.

$\left.\begin{matrix}\cos^m\theta\,d\theta,\\ \sin^n\theta\cos^m\theta\,d\theta,\end{matrix}\right\}$ m odd and $+$, let $\sin\theta=z$, $\therefore\ \cos\theta\,d\theta=dz$.

$\left.\begin{matrix}\sin^m\theta\,d\theta,\ \cos^m\theta\,d\theta,\ m \text{ even and} -,\\ \sin^m\theta\cos^n\theta\,d\theta,\ m+n \text{ even and} -,\end{matrix}\right\}$ let $\tan\theta=z$, $\therefore\ \cos\theta=1/(1+z^2)^{\frac{1}{2}}$, $\sin\theta=z/(1+z^2)^{\frac{1}{2}}$, $d\theta=dz/(1+z^2)$.

104. More generally, any rational function of $\sin\theta$ or $\cos\theta$ becomes algebraic and rational when $\tan\frac{1}{2}\theta = z$. For

$$\cos\theta = (1-z^2)/(1+z^2),\ \sin\theta = 2z/(1+z^2),\ d\theta = 2\,dz/(1+z^2).$$

105. Any rational function of $\tan\theta$ becomes algebraic and rational when $\tan\theta = z$. For $d\theta = dz/(1+z^2)$.

106. Any rational function of e^x becomes algebraic and rational when $e^x = z$. For $dx = dz/z$.

107. On the other hand, certain algebraic surds are rendered trigonometric and rational by substitution. For

if $x = a\sin\theta$, $(a^2-x^2)^{\frac{1}{2}} = a\cos\theta$;
if $x = a\tan\theta$, $(x^2+a^2)^{\frac{1}{2}} = a\sec\theta$;
if $x = a\sec\theta$, $(x^2-a^2)^{\frac{1}{2}} = a\tan\theta$;
if $x = 2a\sin^2\theta$, $(2ax-x^2)^{\frac{1}{2}} = 2a\sin\theta\cos\theta$;
if $x = 2a\tan^2\theta$, $(x^2+2ax)^{\frac{1}{2}} = 2a\sec\theta\tan\theta$;
if $x = 2a\sec^2\theta$, $(x^2-2ax)^{\frac{1}{2}} = 2a\sec\theta\tan\theta$.

Hyperbolic substitutions may also be employed. For

if $x = a\sinh z$, $(x^2+a^2)^{\frac{1}{2}} = a\cosh z$;
if $x = a\cosh z$, $(x^2-a^2)^{\frac{1}{2}} = a\sinh z$;
if $x = 2a\sinh^2 z$, $(x^2+2ax)^{\frac{1}{2}} = 2a\sinh z\cosh z$;
if $x = 2a\cosh^2 z$, $(x^2-2ax)^{\frac{1}{2}} = 2a\sinh z\cosh z$.

108. Since $ax^2+bx+c = \frac{1}{4a}[(2ax+b)^2+4ac-b^2]$, the following general results may be obtained from previous integrations by the substitution $2ax+b=z$.

$$(1)\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)$$

if $4ac-b^2$ is $+$, and

$$= \frac{1}{\sqrt{b^2-4ac}}\log\left(\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right)$$

if b^2-4ac is $+$.

(2) $$\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\log\left[2ax+b+2\sqrt{a(ax^2+bx+c)}\right],$$

$$\int\frac{dx}{\sqrt{-ax^2+bx+c}}=\frac{1}{\sqrt{a}}\sin^{-1}\left(\frac{2ax-b}{\sqrt{4ac+b^2}}\right).$$

(3) $$\int\frac{dx}{(ax^2+bx+c)^{\frac{3}{2}}}=\frac{2(2ax+b)}{(4ac-b^2)\sqrt{ax^2+bx+c}}.$$

(4) $$\int\frac{x\,dx}{ax^2+bx+c}=\frac{1}{2a}\int\frac{(2ax+b)dx-b\,dx}{ax^2+bx+c}$$

$$=\frac{1}{2a}\log\,(ax^2+bx+c)-\frac{b}{2a}\int\frac{dx}{ax^2+bx+c}.$$

(5) $$\int\frac{x\,dx}{\sqrt{ax^2+bx+c}}=\frac{1}{a}\sqrt{ax^2+bx+c}-\frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}}.$$

(6) $$\int\frac{x\,dx}{(ax^2+bx+c)^{\frac{3}{2}}}=-\frac{2(bx+2c)}{(4ac-b^2)\sqrt{ax^2+bx+c}}.$$

109. If we put $x=\frac{1}{z}$ in $\int\frac{dx}{x\sqrt{ax^2+bx+c}}$,

or $x+k=\frac{1}{z}$ in $\int\frac{dx}{(x+k)\sqrt{ax^2+bx+c}}$,

these integrals will be reduced to § 108 (2).

EXAMPLES.

1. $\int\frac{dx}{x\sqrt{2ax-a^2}}=\frac{2}{a}\tan^{-1}\sqrt{\frac{2x}{a}-1}$. Let $\sqrt{2ax-a^2}=z$.

2. $\int\frac{x\sqrt{x}\,dx}{1+x}=(\frac{1}{3}x-1)\,2\sqrt{x}+2\tan^{-1}\sqrt{x}$. Let $\sqrt{x}=z$.

3. $\int\frac{dx}{(2+x)\sqrt{1+x}}=2\tan^{-1}\sqrt{1+x}$.

4. $\int \frac{dx}{(2ax+x^2)^{\frac{3}{2}}} = -\frac{x+a}{a^2\sqrt{2ax+x^2}}$. Let $x+a=\frac{1}{z}$.

5. $\int \frac{dx}{\sqrt{(x-a)(x-b)}} = 2\log(\sqrt{x-a}+\sqrt{x-b})$.

6. $\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2\sin^{-1}\sqrt{\frac{x-a}{b-a}}$.

} Let $x-a=z^2$.

7. $\int x^3(1+x^2)^{\frac{5}{2}}dx = \frac{1}{63}(7x^2-2)(1+x^2)^{\frac{7}{2}}$.

8. $\int \frac{dx}{x^4\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{3x^3}(1+2x^2)$.

9. $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = -\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{\frac{1-x^2}{2x^2}}$.

10. $\int \sin^5\theta\, d\theta = -\frac{1}{5}\cos^5\theta + \frac{2}{3}\cos^3\theta - \cos\theta$.

11. $\int \frac{\sin^3\theta\, d\theta}{\cos^4\theta} = -\frac{1}{\cos\theta} + \frac{1}{3}\frac{1}{\cos^3\theta}$.

12. $\int d\theta \sqrt{\frac{\sin\theta}{\cos^5\theta}} = \frac{2}{3}\sqrt{\tan^3\theta}$.

13. $\int \frac{d\theta}{\sin^4\theta\cos^2\theta} = -\frac{1}{3}\cot^3\theta - 2\cot\theta + \tan\theta$.

14. $\int \frac{d\theta}{1+\sin\theta} = -\frac{2}{1+\tan\frac{1}{2}\theta}$.

15. $\int \frac{\sin\theta - \sin^2\theta}{1+\sin\theta}d\theta = 2\theta + \cos\theta + \frac{4}{1+\tan\frac{1}{2}\theta}$.

16. $\int \frac{d\theta}{a+b\cos\theta} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\tfrac{1}{2}\theta\right)$, $a|>|b$,

and $= \frac{2}{\sqrt{b^2-a^2}}\tanh^{-1}\left(\sqrt{\frac{b-a}{b+a}}\tan\tfrac{1}{2}\theta\right)$, $a|<|b$.

17. $\int \tan^4\theta\, d\theta = \frac{1}{3}\tan^3\theta - \tan\theta + \theta$.

18. $\int \frac{dx}{x^3\sqrt{x^2-1}} = \frac{\sec^{-1}x}{2} + \frac{\sqrt{x^2-1}}{2x^2}.$

19. $\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{c}} \log \frac{x}{bx+2c+2\sqrt{c(ax^2+bx+c)}}.$

20. $\int \frac{dx}{x\sqrt{ax^2+bx-c}} = \frac{1}{\sqrt{c}} \sin^{-1} \frac{bx-2c}{x\sqrt{b^2+4ac}}.$

CHAPTER XXVI.

INTEGRATION BY PARTS.

110. Since $v\,du+u\,dv=d(uv)$, $\therefore\ \int v\,du+\int u\,dv=uv$,

$$\therefore\ \int u\,dv=uv-\int v\,du,$$

or $u\,dv$ can be integrated provided that $v\,du$ can be. Integration by this formula is called integration by parts

Ex. 1. $\int \log x\,dx=(\log x)x-\int x\cdot\frac{dx}{x}$

$$=(\log x)x-\int dx=x\log x-x.$$

2. $\int \sin^{-1}x\,dx=(\sin^{-1}x)x-\int x\,\frac{dx}{\sqrt{1-x^2}}=x\sin^{-1}x+\sqrt{1-x^2}.$

3. $\int x\log x\,dx=\int \log x\,.\,x\,dx=(\log x)\frac{x^2}{2}-\int\frac{x^2}{2}\cdot\frac{dx}{x}$

$$=\frac{x^2}{2}\log x-\frac{x^2}{4}.$$

4. Similarly, $\int x^n\log x\,dx=\frac{x^{n+1}}{n+1}\log x-\frac{x^{n+1}}{(n+1)^2}.$

111. It is often necessary to repeat the integration by parts before the complete integral is obtained.

Ex. $\int x^2 \cos x\, dx = \int x^2 \,.\, \cos x\, dx = x^2 \sin x - 2\int x \sin x\, dx.$

Again, $\int x \sin x\, dx = \int x \,.\, \sin x\, dx = -x \cos x + \int \cos x\, dx$

$$= -x \cos x + \sin x.$$

Substituting, $\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2 \sin x.$

112. Sometimes the integration reproduces the given expression with a new coefficient.

Ex. 1. $\sqrt{a^2-x^2} = \dfrac{a^2-x^2}{\sqrt{a^2-x^2}},$

$$\therefore \int \sqrt{a^2-x^2}\, dx = a^2 \int \frac{dx}{\sqrt{a^2-x^2}} - \int \frac{x^2\, dx}{\sqrt{a^2-x^2}}$$

$$= a^2 \sin^{-1}\frac{x}{a} - \int x\, d(-\sqrt{a^2-x^2})$$

$$= a^2 \sin^{-1}\frac{x}{a} + x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\, dx.$$

Transposing the last term to the left-hand side and dividing by 2, we have

$$\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2} \sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2}. \qquad (1)^*$$

2. Similarly,

$$\int \sqrt{x^2 \pm a^2}\, dx = \pm\frac{a^2}{2} \log\,(x + \sqrt{x^2 \pm a^2}) + \frac{x}{2}\sqrt{x^2 \pm a^2}. \qquad (2)^\dagger$$

---

* (1) and (2) are of frequent occurrence and should be carefully noted. They are also easily obtained by the substitutions of § 107.

† Or,

$$\left.\begin{aligned} \int \sqrt{x^2+a^2}\, dx &= \frac{a^2}{2} \sinh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{x^2+a^2}, \\ \int \sqrt{x^2-a^2}\, dx &= -\frac{a^2}{2} \cosh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{x^2-a^2}. \end{aligned}\right\} \qquad (3)$$

3. $\int \sec\theta \tan^2\theta\, d\theta = \tan\theta \sec\theta - \int \sec^3\theta\, d\theta,$

hence, since $\sec^2\theta = 1 + \tan^2\theta$ and $\int \sec\theta\, d\theta = \log(\sec\theta + \tan\theta)$,

$$\int \sec\theta \tan^2\theta\, d\theta = \tfrac{1}{2}\sec\theta\tan\theta - \tfrac{1}{2}\log(\sec\theta + \tan\theta).$$

EXAMPLES.

1. $\int x\cos x\, dx = x\sin x + \cos x.$

2. $\int \tan^{-1}x\, dx = x\tan^{-1}x - \log\sqrt{1+x^2}.$

3. $\int x\tan^{-1}x\, dx = \tfrac{1}{2}(1+x^2)\tan^{-1}x - \tfrac{1}{2}x.$

4. $\int x\sec^{-1}x\, dx = \tfrac{1}{2}(x^2\sec^{-1}x - \sqrt{x^2-1}).$

5. $\int xe^x dx = (x-1)e^x.$

6. $\int x^2e^x\, dx = (x^2-2x+2)e^x.$

7. $\int e^x\sin x\, dx = \tfrac{1}{2}e^x(\sin x - \cos x).$

8. $\int x^2\sin x\, dx = 2x\sin x + (2-x^2)\cos x.$

9. $\int x\sec^2x\, dx = x\tan x + \log\cos x.$

10. $\int e^x\sin 2x\, dx = \frac{e^x}{5}(\sin 2x - 2\cos 2x).$

11. $\int x\tan^2x\, dx = x\tan x + \log\cos x - \tfrac{1}{2}x^2.$

12. $\int x^2(\log x)^2 dx = \frac{x^3}{3}[(\log x)^2 - \frac{2}{3}\log x + \frac{2}{9}]$.

13. $\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\log(\sec\theta + \tan\theta)$.

14. $\int e^{ax}\cos mx\, dx = \frac{e^{ax}}{a^2+m^2}(a\cos mx + m\sin mx)$.

15. $\int e^{ax}\sin mx\, dx = \frac{e^{ax}}{a^2+m^2}(a\sin mx - m\cos mx)$.

16. $\int \sqrt{2ax-x^2}\, dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)$.

17. $\int \sqrt{2ax+x^2}\, dx = \frac{x+a}{2}\sqrt{2ax+x^2} - \frac{a^2}{2}\log(x+a+\sqrt{2ax+x^2})$.

CHAPTER XXVII.

SUCCESSIVE REDUCTION.

113. To integrate $\sin^n\theta\, d\theta$, n being a positive integer.

$$\int \sin^n\theta\, d\theta = \int \sin^{n-1}\theta \sin\theta\, d\theta$$

$$= -\sin^{n-1}\theta\cos\theta + (n-1)\int \sin^{n-2}\theta\cos^2\theta\, d\theta,$$

and writing $1-\sin^2\theta$ for $\cos^2\theta$,

$$= -\sin^{n-1}\theta\cos\theta + (n-1)\int \sin^{n-2}\theta\, d\theta - (n-1)\int \sin^n\theta\, d\theta.$$

Transposing the last term and dividing by n, we get

$$\int \sin^n\theta\, d\theta = -\frac{\sin^{n-1}\theta\cos\theta}{n} + \frac{n-1}{n}\int \sin^{n-2}\theta\, d\theta. \qquad (1)$$

Writing $n-2$ for n, we have

$$\int \sin^{n-2}\theta\, d\theta = -\frac{\sin^{n-3}\theta\cos\theta}{n-2} + \frac{n-3}{n-2}\int \sin^{n-4}\theta\, d\theta.$$

Thus by each integration the index is diminished by 2, and hence will in the end depend upon $\int \sin\theta\, d\theta = -\cos\theta$, or $\int d\theta = \theta$, according as n is odd or even.

By a similar process

$$\int \cos^n\theta\, d\theta = \frac{\cos^{n-1}\theta \sin\theta}{n} + \frac{n-1}{n}\int \cos^{n-2}\theta\, d\theta. \qquad (2)$$

114. $$\int \frac{d\theta}{\sin^n\theta} = \int \frac{\cos^2\theta + \sin^2\theta}{\sin^n\theta} d\theta = \int \frac{\cos^2\theta}{\sin^n\theta} d\theta + \int \frac{d\theta}{\sin^{n-2}\theta}.$$

$$\text{The first term} = \int \cos\theta\, d\left(-\frac{1}{(n-1)\sin^{n-1}\theta}\right)$$

$$= -\frac{\cos\theta}{(n-1)\sin^{n-1}\theta} - \frac{1}{n-1}\int \frac{d\theta}{\sin^{n-2}\theta}.$$

$$\therefore \int \frac{d\theta}{\sin^n\theta} = -\frac{\cos\theta}{(n-1)\sin^{n-1}\theta} + \frac{n-2}{n-1}\int \frac{d\theta}{\sin^{n-2}\theta},$$

by which we may reduce to

$$\int \frac{d\theta}{\sin\theta} = \log \tan\tfrac{1}{2}\theta, \quad \text{or} \quad \int \frac{d\theta}{\sin^2\theta} = -\cot\theta,$$

according as n is odd or even.

Similarly, $$\int \frac{d\theta}{\cos^n\theta} = \frac{\sin\theta}{(n-1)\cos^{n-1}\theta} + \frac{n-2}{n-1}\int \frac{d\theta}{\cos^{n-2}\theta},$$

and $$\int \frac{d\theta}{\cos\theta} = \log \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta\right), \quad \int \frac{d\theta}{\cos^2\theta} = \tan\theta.$$

115. $$\int \tan^n\theta\, d\theta = \int \tan^{n-2}\theta\, (\sec^2\theta - 1) d\theta$$

$$= \int \tan^{n-2}\theta\, d(\tan\theta) - \int \tan^{n-2}\theta\, d\theta$$

$$= \frac{\tan^{n-1}\theta}{n-1} - \int \tan^{n-2}\theta\, d\theta,$$

and $$\int \tan\theta\, d\theta = \log \sec\theta, \quad \int d\theta = \theta.$$

Similarly, $\int \cot^n\theta\, d\theta = -\frac{\cot^{n-1}\theta}{n-1} - \int \cot^{n-2}\theta\, d\theta,$

and $\int \cot\theta\, d\theta = \log\sin\theta, \quad \int d\theta = \theta.$

116. $\int \cos^m\theta \sin^n\theta\, d\theta = \int \cos^{m-1}\theta\, d\left(\frac{\sin^{n+1}\theta}{n+1}\right)$

$$= \frac{\cos^{m-1}\theta \sin^{n+1}\theta}{n+1} + \frac{m-1}{n+1}\int \cos^{m-2}\theta \sin^{n+2}\theta\, d\theta,$$

and writing $\sin^{n+2}\theta$ in the form $\sin^n\theta\ (1-\cos^2\theta)$,

$$= \frac{\cos^{m-1}\theta \sin^{n+1}\theta}{n+1} + \frac{m-1}{n+1}\int \cos^{m-2}\theta \sin^n\theta\, d\theta - \frac{m-1}{n+1}\int \cos^m\theta \sin^n\theta\, d\theta.$$

Transposing the last term and dividing,

$$\int \cos^m\theta \sin^n\theta\, d\theta = \frac{\cos^{m-1}\theta \sin^{n+1}\theta}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2}\theta \sin^n\theta\, d\theta.$$

In a similar way we might have obtained

$$\int \cos^m\theta \sin^n\theta\, d\theta = -\frac{\cos^{m+1}\theta \sin^{n-1}\theta}{m+n} + \frac{n-1}{m+n}\int \cos^m\theta \sin^{n-2}\theta\, d\theta.$$

117. The following will present no difficulty:

$$\int \frac{\cos^m\theta}{\sin^n\theta}\, d\theta = -\frac{\cos^{m-1}\theta}{(n-1)\sin^{n-1}\theta} - \frac{m-1}{n-1}\int \frac{\cos^{m-2}\theta}{\sin^{n-2}\theta}\, d\theta,$$

$$\int \frac{\sin^n\theta}{\cos^m\theta}\, d\theta = \frac{\sin^{n-1}\theta}{(m-1)\cos^{m-1}\theta} - \frac{n-1}{m-1}\int \frac{\sin^{n-2}\theta}{\cos^{m-2}\theta}\, d\theta.$$

118. $\int \frac{d\theta}{\cos^m\theta \sin^n\theta} = \int \frac{\cos^2\theta + \sin^2\theta}{\cos^m\theta \sin^n\theta} d\theta$

$$\int \frac{d\theta}{\cos^{m-2}\theta \sin^n\theta} + \int \frac{d\theta}{\cos^m\theta \sin^{n-2}\theta}.$$

$$\text{The first term} = \int \frac{1}{\cos^{m-1}\theta} d\left(\frac{-1}{(n-1)\sin^{n-1}\theta}\right)$$

$$= -\frac{1}{(n-1)\cos^{m-1}\theta \sin^{n-1}\theta} + \frac{m-1}{n-1}\int \frac{d\theta}{\cos^m\theta \sin^{n-2}\theta}.$$

Substituting this, we get

$$\int \frac{d\theta}{\cos^m\theta \sin^n\theta} = -\frac{1}{(n-1)\cos^{m-1}\theta \sin^{n-1}\theta} + \frac{m+n-2}{n-1}\int \frac{d\theta}{\cos^m\theta \sin^{n-2}\theta}.$$

By treating the second term in the same way we might have obtained

$$\int \frac{d\theta}{\cos^m\theta \sin^n\theta} = \frac{1}{(m-1)\cos^{m-1}\theta \sin^{n-1}\theta} + \frac{m+n-2}{m-1}\int \frac{d\theta}{\cos^{m-2}\theta \sin^n\theta}.$$

Hence the integration may be reduced to one of the following:

$$\int d\theta, \quad \int \frac{d\theta}{\cos\theta \sin\theta}, \quad \int \frac{d\theta}{\sin\theta}, \quad \text{or} \quad \int \frac{d\theta}{\cos\theta} \text{ (Ch. XXII).}$$

119. The following may be obtained from the preceding reductions by the substitutions of § 107; they may also be obtained directly (cf. § 112).

$$\int \frac{x^n\,dx}{\sqrt{a^2-x^2}} = -\frac{x^{n-1}\sqrt{a^2-x^2}}{n} + \frac{a^2(n-1)}{n}\int \frac{x^{n-2}\,dx}{\sqrt{a^2-x^2}},$$

$$\int \frac{dx}{x^n\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2(n-1)x^{n-1}} + \frac{n-2}{a^2(n-1)}\int \frac{dx}{x^{n-2}\sqrt{a^2-x^2}},$$

$$\int \frac{x^n\,dx}{\sqrt{2ax-x^2}} = -\frac{x^{n-1}\sqrt{2ax-x^2}}{n} + \frac{a(2n-1)}{n}\int \frac{x^{n-1}\,dx}{\sqrt{2ax-x^2}},$$

$$\int \frac{x^n\,dx}{\sqrt{x^2+2ax}} = \frac{x^{n-1}\sqrt{x^2+2ax}}{n} - \frac{a(2n-1)}{n}\int \frac{x^{n-1}\,dx}{\sqrt{x^2+2ax}},$$

$$\int (a^2-x^2)^{\frac{n}{2}}\,dx = \frac{x(a^2-x^2)^{\frac{n}{2}}}{n+1} + \frac{a^2n}{n+1}\int (a^2-x^2)^{\frac{n}{2}-1}\,dx.$$

Examples.

1. Obtain the results of § 114 by integrating $\sec^n\theta\, d\theta$ and $\operatorname{cosec}^n\theta\, d\theta$.

2. $$\int x^m \sin nx\, dx = -\frac{x^m \cos nx}{n} + \frac{mx^{m-1}\sin nx}{n^2} - \frac{m(m-1)}{n^2}\int x^{m-2}\sin nx\, dx.$$

3. $$\int x^m \cos nx\, dx = \frac{x^m \sin nx}{n} + \frac{mx^{m-1}\cos nx}{n^2} - \frac{m(m-1)}{n^2}\int x^{m-2}\cos nx\, dx.$$

4. $$\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx.$$

5. $$\int \frac{e^{ax}\, dx}{x^n} = -\frac{e^{ax}}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{e^{ax}\, dx}{x^{n-1}}.$$

6. $$\int x^m (\log x)^n\, dx = \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1}\, dx.$$

CHAPTER XXVIII.

CERTAIN DEFINITE INTEGRALS.*

120. The first term of § 113 (1) is 0 when $\theta=0$ and also when $\theta=\frac{1}{2}\pi$; hence

$$\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \frac{(n-1)(n-3)\dots}{n(n-2)\dots}\cdot\alpha,$$

each set of factors being carried to 2 or 1, and α being $\frac{\pi}{2}$ when n is even, and 1 when n is odd.

Also, $$\int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta.$$

121. By examining the results of § 116 it will be found that

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{[(m-1)(m-3)\dots][(n-1)(n-3)\dots]}{(m+n)(m+n-2)\dots}\cdot\alpha,$$

each set of factors being carried to 2 or 1, α being $\frac{\pi}{2}$ when m and n are both even, and 1 in all other cases.

This reduces to $\frac{1}{m+1}$ if $n=1$, and to $\frac{1}{n+1}$ if $m=1$.

122. Many integrals may be reduced to the foregoing.

Ex. 1. $\int_0^a \frac{x^n\, dx}{\sqrt{a^2-x^2}} = a^n\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta$. Let $x=a\sin\theta$ (§ 107).

---

* For a collection of indefinite and definite integrals see Peirce's *Short Table of Integrals* (Ginn & Co.).

2. $\int_0^a (a^2-x^2)^{\frac{n}{2}}\,dx = a^{n+1}\int_0^{\frac{\pi}{2}} \cos^{n+1}\theta\,d\theta.$

3. $\int_0^a x^m(a^2-x^2)^{\frac{n}{2}}\,dx = a^{m+n+1}\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^{n+1}\theta\,d\theta.$

4. $\int_0^{2a} \frac{x^n dx}{\sqrt{2ax-x^2}} = 2(2a)^n\int_0^{\frac{\pi}{2}} \sin^{2n}\theta\,d\theta.$

5. $\int_0^{2a} x^m(2ax-x^2)^{\frac{n}{2}}dx = 2(2a)^{m+n+1}\int_0^{\frac{\pi}{2}} \sin^{2m+n+1}\theta\cos^{n+1}\theta\,d\theta.$

6. $\int_0^{\infty} \frac{dx}{(a^2+x^2)^{\frac{n}{2}}} = \frac{1}{a^{n-1}}\int_0^{\frac{\pi}{2}} \cos^{n-2}\theta\,d\theta.$

7. $\int_0^a x^m(a-x)^n\,dx = 2a^{m+n+1}\int_0^{\frac{\pi}{2}} \sin^{2m+1}\theta\cos^{2n+1}\theta\,d\theta.$

123. From § 93 it is plain that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx;$$

that is, interchanging the limits merely changes the sign of the definite integral.

124. It is possible in certain cases to arrive at the value of a definite integral when the indefinite integral is unknown. The following important integral is an illustration.

To prove $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$

From § 120 we have

$$\int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta \,.\, \int_0^{\frac{\pi}{2}} \sin^{n+1}\theta\,d\theta = \frac{1}{n+1}\,\frac{\pi}{2}. \qquad (1)$$

Let $\sin^n\theta = e^{-x^2}$, or $\theta = \sin^{-1}\left(e^{-\frac{x^2}{n}}\right)$.

$$\therefore\ d\theta = \frac{-\frac{2x}{n}e^{-\frac{x^2}{n}}dx}{\sqrt{1-e^{-\frac{2x^2}{n}}}} = -\frac{2x}{n}\frac{dx}{\sqrt{e^{\frac{2x^2}{n}}-1}},$$

which, by the Exponential Theorem,

$$= -\sqrt{\frac{2}{n}}\cdot\frac{dx}{D}, \text{ where } D \equiv \sqrt{1+\frac{x^2}{n}+\frac{2x^4}{3n^2}+\dots}$$

Substituting in (1) we have

$$\int_\infty^0 \sqrt{\frac{2}{n}}\cdot\frac{e^{-x^2}\,dx}{D}\cdot\int_\infty^0 \sqrt{\frac{2}{n}}\cdot\frac{e^{-x^2\left(\frac{n+1}{n}\right)}dx}{D} = \frac{1}{n+1}\frac{\pi}{2},$$

or

$$\int_0^\infty \frac{e^{-x^2}\,dx}{D}\cdot\int_0^\infty \frac{e^{-x^2\left(\frac{n+1}{n}\right)}dx}{D} = \frac{n}{n+1}\frac{\pi}{4}.$$

Now let $n=\infty$; then $D \doteq 1$, and $\frac{n+1}{n}\left(=1+\frac{1}{n}\right)$ also $\doteq 1$,

$$\therefore\ \left[\int_0^\infty e^{-x^2}\,dx\right]^2 = \frac{\pi}{4},\quad \therefore\ \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

EXAMPLES.

1. $\int_0^\pi \sin^n\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta,\quad n \gtrless 0.$

2. $\int_0^\pi \cos^n\theta\,d\theta = 0,$ n odd, and $= 2\int_0^{\frac{\pi}{2}} \cos^n\theta\,d\theta,$ n even

3. $\int_0^\pi x\sin x\cos nx\,dx = (-1)^{n+1}\frac{\pi}{n^2-1},$ n an integer not 1,

and $$= -\frac{\pi}{4} \quad \text{if } n=1$$

4. $\int_0^\pi x \cos x \sin nx \, dx = (-1)^n \frac{\pi n}{n^2-1}$, n an integer not 1,

and $$= -\frac{\pi}{4} \quad \text{if } n=1.$$

5. The Gamma Function. The integral $\int_0^\infty x^{n-1}e^{-x}\,dx$ (n positive) is called the gamma function and is represented by $\Gamma(n)$. Integrate by parts * and show that

$$\Gamma(n)=\frac{1}{n}\Gamma(n+1), \quad \text{or} \quad \Gamma(n+1)=n\Gamma(n).$$

6 Show that $\Gamma(1)=1$, and deduce $\Gamma(2)=1$. $\Gamma(3)=1\,.\,2$, $\Gamma(4)=1\,.\,2\,.\,3, \ldots \Gamma(n)=(n-1)!$ if n is an integer.

7. Show that $\Gamma(\frac{1}{2})=\sqrt{\pi}$. Let $x=z^2$ in the integral. Deduce $\Gamma(n+\frac{1}{2})=1\,.\,3\,.\,5\ldots(2n-1)\sqrt{\pi}/2^n$, n an integer.

8. $\int_0^\infty x^{n-1}e^{-ax}\,dx=\frac{\Gamma(n)}{a^n}$, n and $a>0$

9. $\int_0^\infty x^{2n-1}e^{-x^2}\,dx=\frac{1}{2}\Gamma(n)$, $n>0$. Let $x^2=z$

10. $\int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx=\Gamma(n)$, $n>0$. Let $x=e^{-z}$.

11. $\int_0^1 x^{m-1}\left(\log\frac{1}{x}\right)^{n-1} dx=\frac{\Gamma(n)}{m^n}$, m and $n>0$,

---

* $\mathcal{L}_{x=\infty} x^n e^{-x}=0$ for all values of n.

For, $$x^n e^{-x}=\left(\frac{1}{\frac{1}{x}e^{\frac{x}{n}}}\right)^n=\left(\frac{1}{\frac{1}{x}+\frac{1}{n}+\frac{x}{2n^2}+\ldots}\right)^n \doteq 0 \text{ if } n>0.$$

The conclusion is obvious directly if $n\leqq 0$.

Many other integrals may be expressed by means of gamma functions. The following, known as the Beta Function, is an important illustration (see Williamson's *Integral Calculus*, § 121).

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \quad m \text{ and } n > 0.$$

Assuming this result deduce:

12. $\int_0^a x^{m-1}(a-x)^{n-1}dx = a^{m+n-1}\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$

13. $\int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$ Let $1+x=\frac{1}{z}$.

14. $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}.$

15. $\int_0^{\frac{\pi}{2}} \sin^m\theta\,\cos^n\theta\,d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}, \quad m \text{ and } n > -1.$

Let $\sin^2\theta = z$.

16. $\int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta = \int_0^{\frac{\pi}{2}} \cos^n\theta\,d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$

Values of $\Gamma(n)$ for values of n between 1 and 2 are given at the end of this book. Other values are unnecessary on account of the relation $\Gamma(n+1) = n\,\Gamma(n)$.

CHAPTER XXIX.

AREAS AND LENGTHS OF PLANE CURVES.

SURFACES AND VOLUMES OF SOLIDS OF REVOLUTION.

125. Let P be a point on a continuous curve CD whose equation is given. Let $OA=a$, $OB=b$, $OM=x$, $MP=y$, $MN=dx$, $RQ=\Delta y$, $RT=dy$.

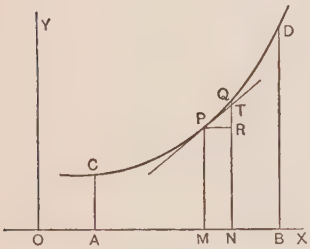

FIG. 66.

We have seen (§§ 94, 95) that

(1) The area $ABDC=\int_a^b y\,dx$.*

(2) The volume formed by the revolution of this area about $OX=\pi\int_a^b y^2\,dx$.

Let the length of the curve measured from some point up to P be s; then $PQ=\Delta s$, $PT=ds$. Also (§§ 92, 93) $\pounds\Sigma\Delta s=\pounds\Sigma ds$. Hence

(3) The length $CD=\int_{x=a}^{x=b} ds$, where $ds=\sqrt{dx^2+dy^2}$.

Δs and ds being equivalent infinitesimals we assume that they form equivalent areas in revolving about the x-axis.

---

* The area $=\sin\omega\int_a^b y\,dx$ if the angle between the axes is ω.

But ds describes the lateral surface of the frustum of a cone whose area

$$=2\pi(y+\tfrac{1}{2}dy)ds=2\pi y\,ds+I,$$

where I is a higher infinitesimal. Hence

(4) The area of the surface formed when CD revolves about OX is $2\pi\int_{x=a}^{x=b} y\,ds$.

Since higher infinitesimals are to be omitted in finding the element of an integral we may use dy for Δy and accordingly regard dx and dy as simultaneous infinitesimal increments of x and y in the same figure. Thus (Fig. 67)

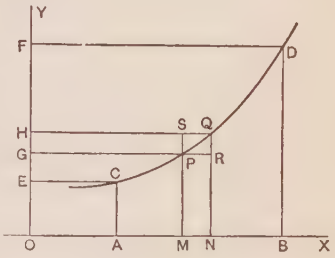

Fig. 67.

(5) The area $ECDF=\int_{OE}^{OF} x\,dy$, and the volume formed by revolving this area about $OY=\pi\int_{OE}^{OF} x^2\,dy$.

If $CABD$ revolves about OY, the volume formed by $MQ=y\,dx\,.\,2\pi x+I$. Hence

(6) The volume formed by revolving $CABD$ about $OY=2\pi\int_a^b xy\,dx$.

(7) Similarly the volume formed by revolving $CABD$ about BD

$$=\int_a^b y\,dx\,.\,2\pi(b-x)=2\pi\int_a^b (b-x)y\,dx.$$

Other relations may be written down in a similar way

Ex. 1. To find the length of the semi-cubical parabola $ay^2 = x^3$ (Fig. 68) from the origin to the point (a, a).

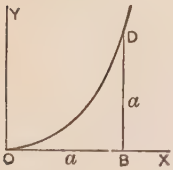

Fig. 68.

From the equation we have $dy = \frac{3}{2\sqrt{a}} x^{\frac{1}{2}}\, dx$,

$$\therefore\ ds = \sqrt{dx^2 + dy^2} = \frac{\sqrt{4a + 9x}}{2\sqrt{a}}\, dx.$$

$$\therefore\ OD = \int_0^a \frac{\sqrt{4a + 9x}}{2\sqrt{a}}\, dx = \left[\frac{(4a + 9x)^{\frac{3}{2}}}{27\sqrt{a}}\right]_0^a$$

$$= \frac{13\sqrt{13} - 8}{27}\, a.$$

2. The area $OBD = \frac{2}{5}a^2$.

3. The volume of OBD about $OX = \frac{1}{4}\pi a^3$.
" " " " " $OY = \frac{4}{7}\pi a^3$.
" " " " " $BD = \frac{8}{35}\pi a^3$.

4. Find the surface of revolution of the cubical parabola $a^2y = x^3$ about OX, x varying from 0 to a. *Ans.* $\frac{\pi}{27}(10\sqrt{10} - 1)a^2$.

126. It is sometimes desirable to express both x and y in terms of a third variable.

Ex. The equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (Fig. 18) is satisfied if we put $x = a \sin^3\theta$, $y = a \cos^3\theta$.

Then $dx = 3a \sin^2\theta \cos\theta\, d\theta$, $\therefore\ \int_0^a y\,dx = 3a^2 \int_0^{\frac{1}{2}\pi} \cos^4\theta \sin^2\theta\, d\theta$,

which, § 121, $= 3a^2 \frac{3}{6.4.2} \frac{\pi}{2} = \frac{3}{32}\pi a^2$.

$\therefore$ the whole area bounded by the curve $= \frac{3}{8}\pi a^2$.

For the length, $ds = \sqrt{dx^2 + dy^2} = 3a \sin\theta \cos\theta\, d\theta$,

$$\therefore\ \text{whole length} = 12a \int_0^{\frac{1}{2}\pi} \sin\theta \cos\theta\, d\theta = 6a.$$

Similarly it may be shown that the volume of the solid made by revolving the whole area about one of the axes $= \frac{32}{105}\pi a^3$, and that the surface of this solid $= \frac{12}{5}\pi a^2$.

127. It will often be necessary to determine the limits of the integration from the equation of the curve. Thus in finding the whole area enclosed by the curve

$$a^2y^2=x^2(a^2-x^2),$$

it will be seen that the curve cuts the x-axis at $(\pm a, 0)$ and that the general shape is that of Fig. 69. Hence the complete area

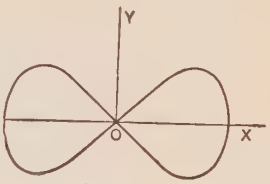

Fig. 69.

$$=4\int_0^a y\,dx=\tfrac{4}{3}a^2.$$

The volume of the solid of revolution about the x-axis $=\frac{4}{15}\pi a^3$, and about the y-axis $=\frac{1}{4}\pi^2a^3$.

128. When y is negative the sign of ydx is $-$, and accordingly an area lying below the axis of x will be affected by the same sign. Hence in calculating an area, care must be taken that y does not change sign between the given limits. Thus in the curve

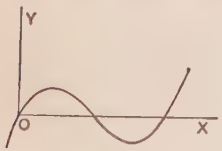

Fig. 70.

$$y=x(x-1)(x-2), \text{ Fig. 70,}$$

y is $+$ from $x=0$ to $x=1$, $-$ from $x=1$ to $x=2$; it will be found that

$$\int_0^1 y\,dx=\tfrac{1}{4}, \quad \int_1^2 y\,dx=-\tfrac{1}{4}, \quad \int_0^2 y\,dx=0.$$

And generally the sum-limit given by a definite integral $\int_a^b f(x)dx$ is that of the algebraical sum of the elements, which will be equal to that of the arithmetical sum only when $f(x)$ is of the same sign for all values of x between a and b.

129. If y is infinite in a given interval of x, the area will have a limit (and will therefore remain finite) if the indefi-

nite integral $\int y\,dx$ remains finite for the interval of x in question. For example, if $y^3(x-1)^2=1$, Fig. 71, $y=\infty$ when $x \doteq 1$, and

$$\int_0^1 y\,dx = 3\Big[(x-1)^{\frac{1}{3}}\Big]_0^1 = 3, \quad \int_1^2 = 3, \quad \int_0^2 = 6.$$

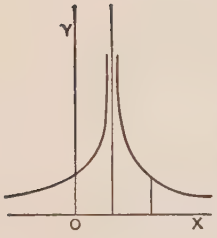

Fig. 71.

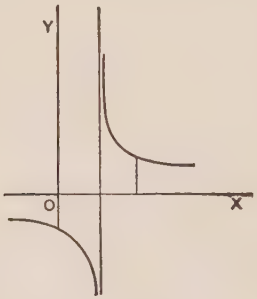

Fig. 72.

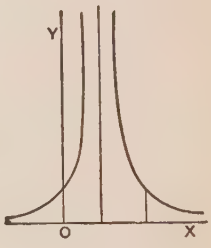

Fig. 73.

Thus if the area is imagined as described by an ordinate which starts from the y-axis and moves towards the asymptote $x=1$, the area $\doteq 3$, and this is what is meant by the area between the curve, the axes, and the asymptote. Similarly if the ordinate starts from the line $x=2$ and moves towards the asymptote $x=1$ the area $\doteq 3$, and the sum of the area-limits $=6=\int_0^2 y\,dx$ as if y were continuous for the interval $[0, 2]$ of x.

Similarly if $y^3(x-1)=1$, Fig. 72,

$$\int_0^1 y\,dx = \tfrac{3}{2}\Big[(x-1)^{\frac{1}{3}}\Big]_0^1 = -\tfrac{3}{2}, \quad \int_1^2 = \tfrac{3}{2}, \quad \int_0^2 = 0,$$

the algebraical sum of the area-limits.

But if $y(x-1)^2=1$, Fig. 73, the indefinite integral $\int y\,dx = \frac{1}{1-x} = \infty$ when $x \doteq 1$, and the area $=\infty$. In this case

$\int_0^2 y\,dx = -2$, which represents no part of the area for the interval [0, 2] of x. On the other hand, the area for $[2, \infty] \doteq 1$.

130. As x increases the volume-element $\pi y^2\,dx$ changes sign only with y^2, i.e., when y becomes imaginary. Thus in Fig. 70,

$$\int_0^2 \pi y^2\,dx = \frac{16\pi}{105} = 2\int_0^1 \pi y^2\,dx.$$

EXAMPLES.

1. The circle $x^2+y^2=a^2$. Show that
   (1) Area $=\pi a^2$.
   (2) Length $=2\pi a$.
   (3) Volume of sphere $=\frac{4}{3}\pi a^3$.
   (4) Surface of sphere $=4\pi a^2$.

2. The witch $y^2(a-x)=a^2x$, Fig. 54.
Let $x=a\sin^2\theta$, then $y=a\tan\theta$.
   (1) Area between curve and asymptote $=\pi a^2$.
   (2) Volume of this about asymptote $=\frac{1}{2}\pi^2 a^3$.
   (3) Volume of same area about $OY=\frac{3}{2}\pi^2 a^3$.

3. The cissoid $y^2(a-x)=x^3$, Fig. 41.
Let $x=a\sin^2\theta$, then $y=a\sin^2\theta\tan\theta$.
   (1) Area between the curve and asymptote $=\frac{3}{4}\pi a^2$.
   (2) Volume of this about asymptote $=\frac{1}{4}\pi^2 a^3$.
   (3) Volume of same area about $OY=\frac{5}{4}\pi^2 a^3$.

4. Find the area bounded by the rectangular hyperbola $xy=1$ and the lines $y=0$, $x=1$, $x=n$. *Ans.* $\log n$.

5. The curve $y^2(a^2-x^2)=a^4$, or $x=a\sin\theta$, $y=a\sec\theta$.
   (1) Area between curve, y-axis, and asymptote $x=a$ is πa^2.
   (2) Volume of this about y-axis $=4\pi a^3$.
   (3) Volume of same area about asymptote $=2\pi a^3(\pi-2)$.

6. The curve $y=e^{-x}$.
   (1) Area from $x=0$ to $x=\infty$ is 1.
   (2) Volume of this about x-axis $=\frac{1}{2}\pi$.
   (3) Convex surface of this solid $=\pi[\sqrt{2}+\log(1+\sqrt{2})]$.

7. The curve $x^2y^2+a^2y^2=a^2x^2$.
The area between the curve and each asymptote $=2a^2$.

8. Find the area between the following curves and the x-axis:

	Ans.
(1) $(y-x)^2=x^3$, Fig. 30.	$\frac{1}{10}$.
(2) $(y-x^2)^2=x^5$, Fig. 31.	$\frac{1}{21}$.
(3) $a^2y=x(x^2-a^2)$, Fig. 17.	$\frac{1}{2}a^2$.
(4) $y(1+x^2)=1$.	π.
(5) $y=x(1-x^2)$.	$\frac{1}{2}$.
(6) $y=x^2(x-1)$.	$\frac{1}{12}$.

9. Find the area of a loop of the curves:

	Ans.
(1) $y^2=x^4(2x+1)$, Fig. 34.	$\frac{4}{105}$.
(2) $y^2=x^2(2x+1)$, Fig. 32.	$\frac{2}{15}$.
(3) $ay^2=(x-a)(x-2a)^2$.	$\frac{8}{15}a^2$.

10. The parabola $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$. See Ex. 11, p. 91.

(1) Area between curve and axes $=\frac{1}{6}ab$.
(2) Volume of this about $OX=\frac{1}{15}\pi ab^2$.

11. The cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$, Fig. 19. For a single arch:

(1) Area $=3\pi a^2$.
(2) Length $=8a$.
(3) Volume about base $=5\pi^2a^3$.
(4) Surface of this solid $=\frac{64}{3}\pi a^2$.
(5) Volume of the area $3\pi a^2$ about tangent at vertex $=7\pi^2a^3$.
(6) Show that in Fig. 20, $s^2=8ax$ ($s=OP$, $x=OM$).

12. The curve $x=a(1-\cos\theta)$, $y=a\theta$; Fig. 20.

(1) Area $=2\pi a^2$.
(2) Volume of this about $OX=\pi(\pi^2-4)a^3$.
(3) Volume about $OY=5\pi^2a^3$.

13. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, or $x=a\sin\theta$, $y=b\cos\theta$. Show that

(1) Area $=\pi ab$.
(2) Volume of prolate spheroid $*=\frac{4}{3}\pi ab^2$.

---

* The solid formed by the revolution of an ellipse about its major axis.

(3) Surface of prolate spheroid $= 2\pi b^2 + \dfrac{2\pi ab}{e}\sin^{-1}e$.

(4) Volume of oblate spheroid * $= \frac{4}{3}\pi a^2 b$.

(5) Surface of oblate spheroid $= 2\pi a^2 + \dfrac{\pi b^2}{e}\log\left(\dfrac{1+e}{1-e}\right)$.

NOTE.—The eccentricity $e = \sqrt{a^2-b^2}/a$.

14. The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, or $x = a\sec\theta$, $y = b\tan\theta$.

Show that the area bounded by the curve, the x-axis, and the ordinate at the point (x_1, y_1) is

$$\tfrac{1}{2}x_1y_1 - \tfrac{1}{2}ab\log\left(\frac{x_1}{a} + \frac{y_1}{b}\right),$$

and hence that the second term in this result is the area of the hyperbolic sector OAP, where O is the centre, A the vertex, and P the point on the curve.

15. The parabola $y^2 = 4ax$, Fig. 74.

If $OA = x_1$, $AB = y_1$, show that

(1) Area $OAB = \frac{2}{3}x_1y_1$.

(2) Length $OB = \dfrac{y_1}{4a}\sqrt{4a^2+y_1^2}$

$$+ a\log\frac{y_1 + \sqrt{4a^2+y_1^2}}{2a}.$$

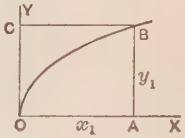

FIG. 74.

(3) Volume of OAB about $OX = \frac{1}{2}\pi y_1^2 x_1$.

(4) Surface of this solid $= \dfrac{\pi}{3a}[(4a^2+y_1^2)^{\frac{3}{2}} - 8a^3]$

$$= \frac{\pi}{3a}[\text{normal}^3 - \text{subnormal}^3].$$

(5) Volume of OAB about $AB = \frac{8}{15}\pi x_1^2 y_1$.

(6) Volume of OBC about $OY = \frac{1}{5}\pi x_1^2 y_1$.

(7) Volume of OBC about $BC = \frac{1}{6}\pi y_1^2 x_1$.

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* The solid formed by the revolution of an ellipse about its minor axis.

CHAPTER XXX.

SIMPSON'S RULE. VOLUMES FROM PARALLEL SECTIONS. THE PRISMOIDAL FORMULA. LENGTH OF A CURVE IN SPACE.

131. Simpson's rule. An area (Fig. 75) is bounded by a line which is taken as the x-axis, a curve, and two ordi-

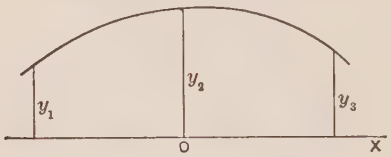

FIG. 75

nates of length y_1, y_3, at a distance h apart, and y_2 is the ordinate midway between them.

The area

$$A=\tfrac{1}{6}h(y_1+4y_2+y_3), \tag{1}$$

provided that the equation of the curve is of the form

$$y=a+bx+cx^2+dx^3, \tag{2}$$

where a, b, c, and d are constants. (1) is the statement of Simpson's Rule.

For convenience take the origin at 0, the foot of the middle ordinate. Then the area

$$A=\int_{-\frac{1}{2}h}^{\frac{1}{2}h} y\,dx=\int_{-\frac{1}{2}h}^{\frac{1}{2}h}(a+bx+cx^2+dx^3)dx$$
$$=ah+\tfrac{1}{12}ch^3=\tfrac{1}{6}h(6a+\tfrac{1}{2}ch^2).$$

y_1, y_2, y_3 are the values of y in (2) when $x=-\frac{1}{2}h$, 0, $\frac{1}{2}h$; $\therefore\ y_1+y_3=2a+\frac{1}{2}ch^2$, and $y_2=a$. Hence (1).

The origin may be any point in OX, for the equation would remain of the form (2) if the origin were transferred to O.

132. When the equation is not of the form (2), or is altogether unknown, the area may be divided into four, six, or an even number n of parts by equidistant ordinates, and (1) applied to each part; the result will be a more or less close approximation to the correct area. This depends upon the fact that y can, in general, be expressed as a series of powers of x, and that higher powers than the third may, for purposes of approximation, be neglected if x is small. Formula (1) now becomes

$$\frac{h}{3n}[y_1+4(y_2+y_4+\dots)+2(y_3+y_5+\dots)+y_{n+1}],$$

h being the whole base, n the number of parts, y_1 and y_{n+1} the extreme ordinates, y_2, y_4, ... the even-numbered ordinates, y_3, y_5, ... the remaining ordinates.

Ex. If in Fig. 75 the base h were divided into *three* equal parts, show that the area

$$=\tfrac{1}{8}h(y_1+3y_2+3y_3+y_4)\ *,$$

where y_1 and y_4 are the extreme ordinates, y_2 and y_3 the intermediate ones.

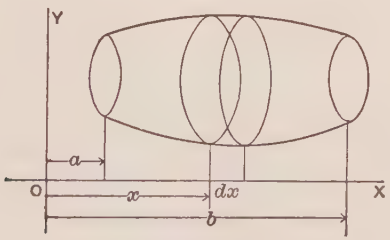

Fig. 76.

133. Volumes from parallel sections. Let a solid be cut by parallel planes at perpendicular distances a, x, $x+dx$, b

* Another of Simpson's Formulæ.

from a fixed point, and let A be the area of the section at distance x. Then if A can be expressed as a function of x, the volume of the solid between the extreme planes is

$$\int_a^b A\,dx.$$

For the volume of the slice of thickness dx is $(A+i)\,dx$, where i is infinitesimal, $\therefore$ the element of the integral is $A\,dx$.

Ex. 1. To find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. The equation may be written

$$\frac{y^2}{b^2\left(1-\frac{x^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\frac{x^2}{a^2}\right)}=1,$$

which, x being regarded as constant, is the equation of a section at a distance x from the origin. The area of any ellipse $y^2/\alpha^2+z^2/\beta^2=1$ is $\pi\alpha\beta$. Hence the area of the section of which DEF is a quadrant is $\pi bc\left(1-\frac{x^2}{a^2}\right)$. $\therefore$ the whole volume is

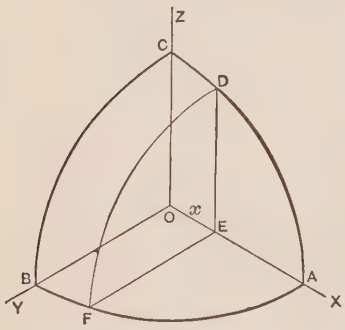

Fig. 77.

$$2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx=\tfrac{4}{3}\pi abc.$$

2. Find the volume of the elliptic paraboloid $y^2/b^2+z^2/c^2=2x$ from $x=0$ to $x=a$. *Ans.* πa^2bc.

3. Find the volume enclosed by the plane $x=h$ and the surface (1) $y^2/x^2+z^2/a^2=1$, (2) $xy^2+az^2=ax^2$.

Ans. (1) $\frac{1}{2}\pi ah^2$, (2) $\frac{2}{5}\pi a^{\frac{1}{2}}h^{\frac{5}{2}}$.

4. Find the volume of the tetrahedron formed by the coördinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. *Ans.* $\frac{1}{6}abc$.

5. Two cylinders of altitude h have one extremity, viz., a circle of radius a, in common; the opposite extremities touch each other. To find the common volume.

A section of the common volume parallel to the plane $CDEF$ (which contains the centres of the circles) and at a distance $OA = x$ from that plane is a triangle GBH similar to EQF. The area of EQF is ah.

$$\therefore \frac{GBH}{ah} = \frac{AH^2}{OF^2} = \frac{a^2 - x^2}{a^2}. \quad \therefore GBH = \frac{h}{a}(a^2 - x^2).$$

$$\therefore V = 2\frac{h}{a}\int_0^a (a^2 - x^2)\, dx = \frac{4}{3}a^2 h.$$

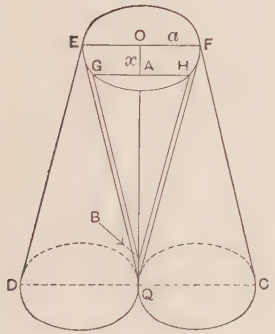

Fig. 78.

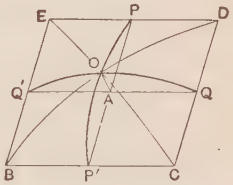

Fig. 79.

6. A square moves with the middle points of its sides on the circumferences of two equal circles at right angles to each other. To show that the volume and surface of the groin thus formed are each $4/\pi$ times those of the inscribed sphere.

Let $BCDE$ (Fig. 79) be one position of the square, $OA = x$, $AP = y$. The volume and surface elements of the sphere are $\pi y^2 dx$, $2\pi y\, ds$; those of the given solid are $(2y)^2 dx$, $4(2y) ds$; hence the proposition. The volume and surface are therefore $\frac{16}{3}a^3$, $16a^2$.

The solid is evidently the common part of two equal right circular cylinders whose axes intersect at right angles.

7. A right circular cylinder is sharpened to an edge coinciding with a diameter, the equal plane faces forming a wedge. Find the volume cut off.

Let a length h be cut from opposite sides of the cylinder of radius a. Sections may be made by planes parallel to the axis and the diameter, or parallel to the axis and perpendicular to the diameter. *Ans.* $\frac{4}{3}a^2h$.

Show that for any diameter of a right elliptic cylinder the result is $\frac{4}{3}abh$.

8. A parallelogram moves with its angular points on two ellipses which have a common axis. The semi-axes are a, b, c, and the angle between the curves is ω Show that the volume is $\frac{8}{3}abc \sin \omega$.

9. Show that the volume of any cone or pyramid $= \frac{1}{3}$ base $\times$ height, assuming that the area of a section parallel to the base varies as the square of its distance from the vertex.

10. A straight line is parallel to a plane which contains a closed curve. Another straight line moves so as to intersect the curve and the fixed straight line and remain perpendicular to the latter. Show that the volume of the *right conoid* thus formed $=$ $\frac{1}{2}$ base $\times$ height.

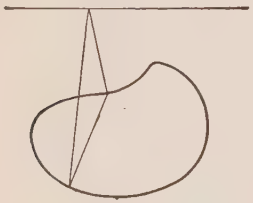

Fig. 80.

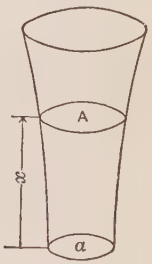

Fig. 81.

11. *Form of an inverted column of uniform strength.* Let A be area of a horizontal section at a distance x above the base, which is also assumed to be horizontal and of area a. The prescribed condition is that A varies as the volume V below A; hence dA varies as dV.

$$\therefore\ dA = kA\ dx, \quad \text{or} \quad dA/A = k\ dx.$$

Integrating, $\log A = kx + c$. But $A = a$ when $x = 0$, $\therefore \log a = c$.

$$\therefore\ \log (A/a) = kx, \quad \text{or} \quad A = ae^{kx}.$$

12. Such a column is to be cast in the form of a solid of revo-

lution, R and r being the radii of the extremities, and h the height. How much metal is required? *Ans.* Vol. $= \dfrac{\pi h(R^2-r^2)}{2\log\left(\dfrac{R}{r}\right)}$.

134. The prismoidal formula. The extremities of a solid are parallel planes of area A_1, A_3, at a distance h apart, and A_2 is the area of a parallel section midway between them. The volume

$$v = \tfrac{1}{6}h(A_1 + 4A_2 + A_3), \tag{1}$$

provided that the area A of any section parallel to the extremities can be expressed in the form

$$a + bx + cx^2 + dx^3, \tag{2}$$

where x is the distance of the section from a fixed point.

(1) is the Prismoidal Formula. Since the volume $= \int A\,dx$, the proof is the same as for Simpson's Rule. The Prismoidal Formula will give exact values of the volume of many of the common solids, such as cones, pyramids, prisms, spheres, ellipsoids, paraboloids, etc. It will apply to Exs. 1–9 of § 133. (In Ex. 7 it will apply to the second mentioned sections, but not to the first.)

Ex. 1. The area of a section of a sphere at a distance x from the centre is $\pi(a^2-x^2)$, which is of the form (2), hence the Prismoidal Formula will apply. For the whole volume $h=2a$, $A_1=A_3=0$, $A_2=\pi a^2$. $\therefore\ V=\frac{4}{3}\pi a^3$.

2. Find the volume of the greatest solid that can be cut from a sphere of radius a, the parallel sections to be regular polygons of n sides. *Ans.* $\frac{2}{3}na^3 \sin 2\pi/n$.

The volume of a sphere may be deduced. For

$$\tfrac{2}{3}na^3 \sin 2\pi/n = \tfrac{4}{3}\pi a^3 \frac{\sin 2\pi/n}{2\pi/n} \doteq \tfrac{4}{3}\pi a^3 \text{ when } n=\infty\,.$$

135. Length of a curve in space.

Ex. 1. To find the length of the curve of intersection of $az = x^2$ and $3a^2y = 2x^3$ from the origin to the point (x_1, y_1, z_1).

We have (§ 66) $ds^2 = dx^2 + dy^2 + dz^2 = \left(1 + \frac{2x^2}{a^2}\right)^2 dx^2$.

$$\therefore\ s = \int_0^{x_1} \left(1 + \frac{2x^2}{a^2}\right) dx = x_1 + \frac{2}{3}\frac{x_1^3}{a^2} = x_1 + y_1.$$

2. Find the length of the helix $x = a \sin nz$, $y = a \cos nz$, from the origin to the point (x_1, y_1, z_1). *Ans.* $z_1\sqrt{1+n^2a^2}$.

CHAPTER XXXI.

POLAR COÖRDINATES.

136. Let O be the pole or polar origin, OA the polar axis or initial line, (θ, r), $(\theta+\Delta\theta, r+\Delta r)$ the coördinates of P and Q, ψ the angle which the tangent at P makes with the radius vector OP. Take PR perpendicular to OQ. Then ψ is the limit of OQP as Q approaches coincidence with P, and $\tan OQP = PR/RQ$.

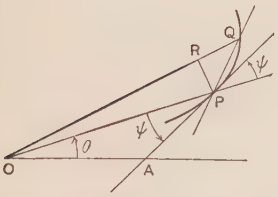

FIG. 82.

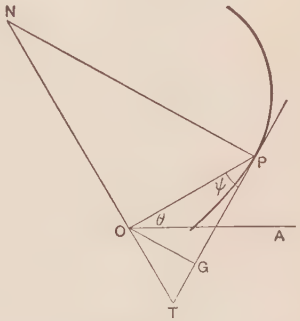

FIG. 83.

But $PR = r\sin\Delta\theta = r\Delta\theta + I_1$, (§ 16), and
$RQ = r+\Delta r - r\cos\Delta\theta = \Delta r + r(1-\cos\Delta\theta) = \Delta r + I_2$.

$$\therefore \tan\psi = \pounds\frac{PR}{RQ} = \pounds\frac{r\Delta\theta}{\Delta r} \text{ (§ 17) } = r\frac{d\theta}{dr}.$$

Similarly $\sin\psi = r\dfrac{d\theta}{ds}$, $\cos\psi = \dfrac{dr}{ds}$.

Squaring and adding, $1=(r^2d\theta^2+dr^2)/ds^2$,

or $$ds^2=r^2d\theta^2+dr^2.$$

137. Through the origin O (Fig. 83) let a line be drawn perpendicular to the radius vector OP, meeting the tangent in T and the normal in N. Then TP is called the *polar tangent*, NP the *polar normal*, TO the *polar subtangent*, and ON the *polar subnormal*. The lengths of these lines in terms of r and θ can be written down at once; e.g.,

$$TO=r\tan\phi=r^2d\theta/dr.$$

The tangent at any point P is easily drawn by calculating OT and then joining T to P.

138. Some of these quantities are more conveniently expressed in terms of θ and the reciprocal of r. Calling this u, we have $u=1/r$, $du=-dr/r^2$; hence the polar subtangent

$$TO=-d\theta/du.$$

The polar coördinates of T are $\left(\frac{1}{2}\pi+\theta, \frac{d\theta}{du}\right)$.

Let OG, the perpendicular on the tangent, $=p$.

Then $\because$ OTP is a right-angled triangle and OG the perpendicular from the right angle to the hypotenuse, we have

$$\frac{1}{OG^2}=\frac{1}{OP^2}+\frac{1}{OT^2}, \quad \text{or} \quad \frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2. \tag{1}$$

139. The polar equations of some of the commoner curves are as follows:

(1) $r\cos\theta=a$, a straight line.

(2) $r=a\cos\theta$, a circle of diameter a (origin a point on the circumference, initial line a diameter).

(3) $r^2\cos 2\theta=a^2$, a rectangular hyperbola, Fig. 98 (origin the centre, initial line the transverse axis).

(4) $r^2=a^2\cos 2\theta$, a lemniscate, Fig. 27 (origin the centre initial line the axis).

(5) $r^{\frac{1}{2}} \cos \frac{1}{2}\theta = a^{\frac{1}{2}}$, or $r(1+\cos\theta)=2a$, a parabola (origin the focus, initial line the axis).

(6) $r^{\frac{1}{2}}=a^{\frac{1}{2}} \cos \frac{1}{2}\theta$, or $r=\frac{1}{2}a(1+\cos\theta)$, a cardioid, Fig. 89.

(7) $r(1+e\cos\theta)=m$, an ellipse, hyperbola, or parabola according as the eccentricity $e<$, $=$, or >1 (pole the focus, initial line the axis, m half the latus rectum).

(8) $r=n(1+e\cos\theta)$, a limaçon, Figs. 88, 89, 90, according as $e<$, $=$, or >1.

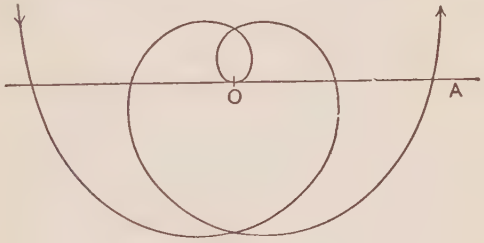

Fig. 84.

(9) $r=a\theta$, a spiral of Archimedes, Fig. 84.

(In Figs. 84, 85, θ varies from a little less than -2π to a little more than 2π).

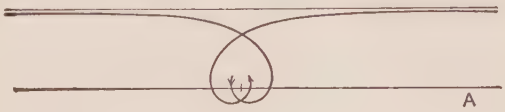

Fig. 85.

(10) $r\theta=a$, a reciprocal or hyperbolic spiral, Fig. 85.

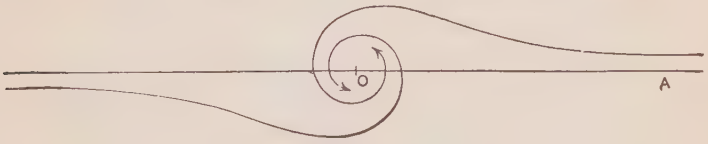

Fig. 86.

(11) $r^2\theta=a^2$, a lituus, Fig. 86 (θ is necessarily $+$, and varies in the figure from 0 to a little more than 2π, r is $\pm$ for a given value of θ).

(12) $r=a^\theta$, a logarithmic or equiangular spiral, Fig. 87 ($r=1$ when $\theta=0$, $r=a$ when $\theta=1$ radian, $r<1$ when θ is negative).

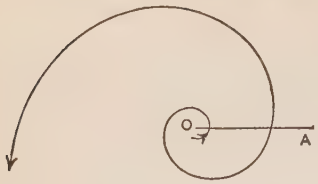

Fig. 87.

Since θ may be supposed to increase or decrease without bound, each spiral consists of an infinite number of whorls or spires.

140. Equations (1) to (6) are all included under the form $r^m \cos m\theta = a^m$; in (1), (3), and (5) m has the values 1, 2, $\frac{1}{2}$, respectively; in (2), (4), (6), it has the values -1, -2, $-\frac{1}{2}$. In all cases a is the intercept on the initial line. The equation $r^m \sin m\theta = a^m$ represents the same series of curves, the initial line having been turned backward through the angle $\pi/(2m)$. Similarly (9), (10), (11) are particular cases of the equation $r^m = a^m\theta^n$.

141. The radius vector of the limaçon, equation (8), is proportional to the reciprocal of the radius vector of a conic section, equation 7; hence the limaçon is called the inverse of a conic section with regard to a focus. Since $r = en\cos\theta + n$, the radius vector is equal to that of a circle of diameter en plus a constant line n, and hence the curve is easily constructed. (The construction or auxiliary circles are shown in the figures.)

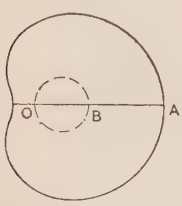

Fig. 88.

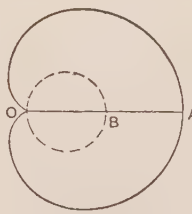

Fig. 89.

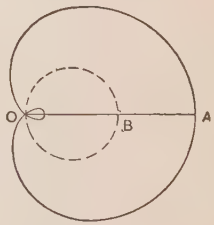

Fig. 90.

When $e=1$ the curve becomes a cardioid (eqn. 6), which is therefore the inverse of a parabola. When $e=2$ the curve is

called a trisectrix, the loop then passing through the centre of the circle.

Examples.

1. If $r^m = a^m \theta^n$, show that $\tan \psi = \frac{m}{n}\theta$.

(Differentiate logarithmically.)

2. If $r^m \cos m\theta = a^m$, or $r^m = a^m \cos m\theta$, show that $\tan \psi = \cot m\theta$, i.e., that the angle between the radius vector and normal $= m\theta$, and hence that GOA (Fig. 83) $= (m-1)\theta$.

3. In the logarithmic spiral $r = a^\theta$ show that ψ is constant and $= \cot^{-1}(\log_e a)$.

In Fig. 87. $a = 1{\cdot}318$ cm.; show that $\psi = 74° 33'$.

4. To find the polar subtangent of a conic.

From the equation $1 + e\cos\theta = m/r = mu$ we have $-e\sin\theta\, d\theta = m\,du$, and the polar subtangent $= -d\theta/du = m/(e\sin\theta)$.

5. In any conic prove that

$$\frac{1}{p^2} = \frac{2}{m}\left(\frac{1}{r} - \frac{1-e^2}{2m}\right).$$

6. In the curve $r^m \cos m\theta = a^m$ prove that $pr^{m-1} = a^m$.

7. Changing the sign of m, show that $pa^m = r^{m+1}$ in the curve $r^m = a^m \cos m\theta$.

8. Show that the polar subnormal of any curve $= dr/d\theta$.

In what curve is the polar subnormal constant?

9. In what curve is the polar subtangent constant?

10. Show that the polar normal $= ds/d\theta$.

Asymptotes.

142. The position of any line is known when its direction and one point in the line are known We may therefore determine an asymptote by finding a value of θ for which $r = \infty$ or $u = 0$, and then calculating the coördinates (§ 138) of T, the extremity of the corresponding polar subtangent, viz. $\left(\frac{1}{2}\pi + \theta, \frac{d\theta}{du}\right)$, remembering that the asymptote and radius vector must be parallel.

EXAMPLES.

1. $r=\dfrac{a\theta}{1-\theta}$ (Fig. 91), or $u=\dfrac{1}{a\theta}-\dfrac{1}{a}$ whence $\dfrac{d\theta}{du}=-a\theta^2$, and $r=\infty$ or $u=0$ when $\theta=1$. Hence the asymptote passes through the point $(\frac{1}{2}\pi+1,\ -a)$ or $(1-\frac{1}{2}\pi,\ a)$ and is parallel to the line $\theta=1$.

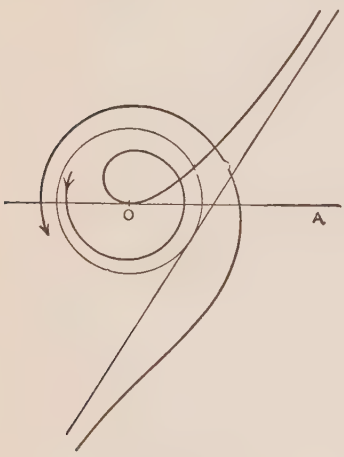

FIG. 91.

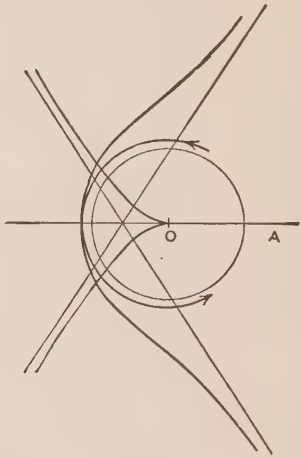

FIG. 92.

2. Find the asymptotes of the curve $(r-a)\theta^2=r$ (Fig. 92).

Ans. Lines through $(\frac{1}{2}\pi\pm1,\ \pm\frac{1}{2}a)$ parallel to $\theta=\pm1$.

3. Find the asymptote of the reciprocal spiral $r\theta=a$ (Fig. 85).

Ans. A line through $(\frac{1}{2}\pi,\ a)$ parallel to the initial line.

4. Show that the initial line is an asymptote to the lituus $r^2\theta=a^2$ (Fig. 86).

5. Find the asymptotes of the curve $r\sin 4\theta=a$ (Fig. 93).

Ans. Four pairs of parallel lines, each pair $\frac{1}{2}a$ apart.

(The numbers in figures indicate the order in which the branches are formed as θ increases from 0 to 2π.)

6. Find the asymptotes of the curve $r^2\sin 4\theta=a^2$ (Fig. 94).

Ans. Four lines passing through the origin.

7. Find the asymptotes of the curve $r\cos 2\theta=2a$.

Ans. Four lines parallel respectively to $\theta=\frac{1}{4}\pi$, $\theta=\frac{3}{4}\pi$, $\theta=\frac{5}{4}\pi$, $\theta=\frac{7}{4}\pi$, and passing through the points $(\frac{3}{4}\pi,\ -a)$, $(\frac{5}{4}\pi,\ a)$, $(\frac{7}{4}\pi,\ -a)$, $(\frac{9}{4}\pi,\ a)$.

8. Show that the rectangular equation of an asymptote of the curve $r^{-1}=f(\theta)$ is

$$f'(\alpha)(x \sin \alpha - y \cos \alpha)+1=0,$$

where α is one of the roots of the equation $f(\theta)=0$.

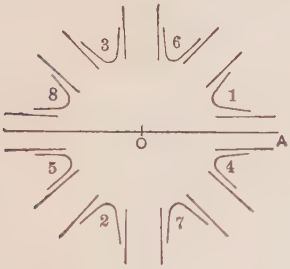

FIG. 93.

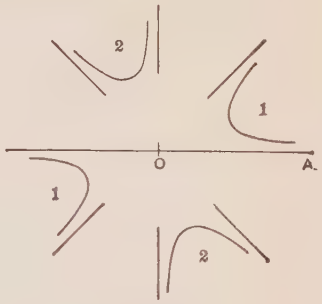

FIG. 94.

Asymptotic Circles.

143. In the curve Fig. 91, $r=\dfrac{a\theta}{1-\theta}=\dfrac{a}{\dfrac{1}{\theta}-1}\doteq -a$ if $\theta=\pm\infty$,

and hence the circle of radius a is called an asymptotic circle. The curve approaches the circle from the outside when θ increases from the value 1, and from the inside when θ decreases from the value 0. Similarly $r=a$ is an asymptotic circle of the curve $r(\theta^2-1)=a\theta^2$ (Fig. 92).

Points of Inflexion.

144. Whenever the extremity of the radius vector passes through a point of inflexion, the perpendicular on the tangent is a maximum or a minimum, and hence dp/dr changes sign.

Differentiating $\dfrac{1}{p^2}=u^2+\left(\dfrac{du}{d\theta}\right)^2$ (§ 138) we have

$$-\frac{2}{p^3}dp=2u\,du+\frac{2\,du\,d^2u}{d\theta^2}=2\,du\left(u+\frac{d^2u}{d\theta^2}\right).$$

Also $r=1/u$, $dr=-du/u^2$,

$$\therefore \frac{dp}{dr}=u^2p^3\left(u+\frac{d^2u}{d\theta^2}\right).$$

Hence at a point of inflexion $u+\dfrac{d^2u}{d\theta^2}$ changes sign.

Examples.

1. Find the points of inflexion on the curve $(r-a)\theta^2=r$ or $au=1-\theta^{-2}$ (Fig. 92). *Ans.* $(\pm\sqrt{3}, \frac{2}{3}a)$.

2. Find the point of inflexion on the curve $r(1-\theta)=a\theta$ (Fig. 91).

Ans. θ is a root of the equation $\theta^3-\theta^2-2=0$, $\therefore$ (§ 50) $\theta=1.696$ rdn. $=97°{\cdot}2$, and $\therefore$ $r=-2{\cdot}437a$.

3. Find the points of inflexion on the lituus $r^2\theta=a^2$ (Fig. 86).

Ans. $(\frac{1}{2}, \pm a\sqrt{2})$

4. In the lemniscate $r^2=a^2\cos 2\theta$ (Fig. 27) show that $dp/dr=3\cos 2\theta$, and hence that the origin is a point of inflexion on each branch.

5. Show that a curve is concave or convex to the origin according as $u+d^2u/d\theta^2$ is + or −.

Multiple Points.

145. The equation of a curve being $r=f(\theta)$, the direction of the curve at the origin is determined by the values of θ,

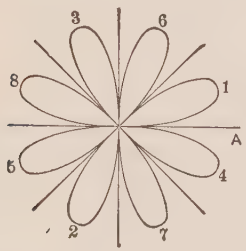

Fig. 95.

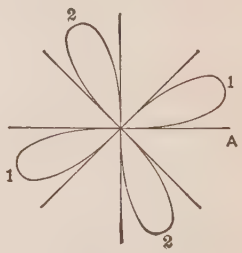

Fig. 96.

which satisfy the equation $f(\theta)=0$. If this equation have two or more roots there will be a multiple point at the origin.

EXAMPLES.

1. In the lemniscate $r^2 = a^2 \cos 2\theta$ (Fig. 27) the equation $\cos 2\theta = 0$ gives $\theta = \pm \frac{1}{4}\pi$ for the directions of the tangents at the origin.

2. Find the tangents to the curve $r = a \sin 4\theta$ (Fig. 95) at the origin. *Ans.* $\theta = 0, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi$.

These lines are also tangents to the curve $r^2 = a^2 \sin 4\theta$ (Fig. 96) at the origin.

3. Find the tangents to the curve $r = a \sin 3\theta$ at the origin.

Ans. $\theta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi$.

4. Show that the curve $(r - a)\theta^2 = r$ (Fig. 92) has a cusp at the origin.

Curvature.

146. Let PD, QD be consecutive normals (see § 84), and let the angle $PDQ = \Delta\phi$. We shall first show that $DP - DQ$ is an infinitesimal of at least the second order, PQ or $\Delta\phi$ being of the first.

Draw QF perpendicular to DP. Then

$$DP - DQ = FP - DQ(1 - \cos \Delta\phi).$$

But $1 - \cos \Delta\phi$ is of the second order; so is FP, since it = chord $PQ \times \cos FPQ$, and each factor is infinitesimal. Hence DP–DQ is of at least the second order.

Let $PD = n$, then QD may be written $n + I$.

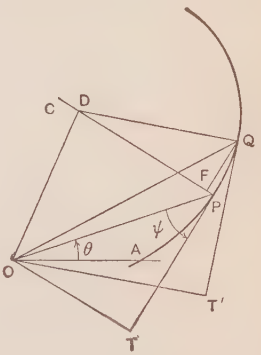

FIG. 97.

Let $OP = r$, $OQ = r + \Delta r$, $OT = p$, $OT' = p + \Delta p$. Then in the triangle OPD

$$OD^2 = PO^2 + PD^2 - 2PO \cdot PD \cos OPD$$
$$= r^2 + n^2 - 2rn \sin \phi = r^2 + n^2 - 2pn.$$

Hence in the triangle OQD

$$OD^2=(r+\Delta r)^2+(n+I)^2-2(p+\Delta p)(n+I).$$

Equating and simplifying,

$$0=2r\,\Delta r-2n\,\Delta p+I_1, \quad \therefore\ \pounds n=\pounds(r\,\Delta r/\Delta p).$$

But $\pounds n=R$, the radius of curvature PC, § 84.

$$\therefore\ R=r\frac{dr}{dp}. \tag{1}$$

Hence also (§§ 138, 144)

$$R=\frac{1}{u^3p^3\left(u+\dfrac{d^2u}{d\theta^2}\right)}=\frac{\left[u^2+\left(\dfrac{du}{d\theta}\right)^2\right]^{\frac{3}{2}}}{u^3\left(u+\dfrac{d^2u}{d\theta^2}\right)},$$

or

$$R=\frac{\left[1+\dfrac{1}{u^2}\left(\dfrac{du}{d\theta}\right)^2\right]^{\frac{3}{2}}}{u+\dfrac{d^2u}{d\theta^2}}. \tag{2}$$

Examples.

1. Find the radius of curvature at any point of $r^m\cos m\theta=a^m$ or $pr^{m-1}=a^m$.

$$Ans.\ R=-\frac{r^{m+1}}{(m-1)a^m}=-\frac{r^2}{(m-1)p}.$$

If $r^m=a^m\cos m\theta$, $R=\dfrac{a^m}{(m+1)r^{m-1}}=\dfrac{r^2}{(m+1)p}$.

2. The equation $r^2=p^2+a^2$ represents an involute of a circle, find R.

3. In the logarithmic spiral $r=a^\theta$, $p=r\sin\psi$, and ψ is constant, hence $R=r/\sin\psi=$ the polar normal.

4. Show that the evolute of the logarithmic spiral is an equal logarithmic spiral.

[OC is a radius vector and PC a tangent to the evolute, and in this case the angle $OCP=\psi$, a constant.]

5. Prove that in any curve

$$R=\frac{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}.$$

[We have $u=1/r$, $du=-dr/r^2$, $d^2u=-(r^2d^2r-2r\,dr^2)/r^4$, to substitute in (2).]

6. In the spiral $r=a\theta$ (Fig. 84), $R=(a^2+r^2)^{\frac{3}{2}}/(2a^2+r^2)$.

7. In the spiral $r\theta=a$ (Fig. 85), $R=r(a^2+r^2)^{\frac{3}{2}}/a^3$.

8. If a curve touch the initial line at the origin, prove that $R=$ the limit of $\frac{1}{2}r/\theta$ at that point; and hence show that the radius of curvature of the curve of Fig. 91 at the origin is half the radius of the circle in the figure.

9. Find R for the curve $r=a\sin n\theta$ at the origin. *Ans.* $\frac{1}{2}na$.

10. Prove that the intercept of the circle of curvature on the radius vector of any curve $=2p\,dr/dp$.

In the curves $r^m\cos m\theta=a^m$ and $r^m=a^m\cos m\theta$ show that these chords $=-2r/(m-1)$ and $2r/(m+1)$, respectively.

Areas, etc.

147. Let $AOP=\theta$, $POQ=d\theta$, $OP=r$, $OQ=r+\Delta r$. (1) The area-increment POQ lies between the circular sectors POD, EOQ, whose areas are $\frac{1}{2}r^2d\theta$, $\frac{1}{2}(r+\Delta r)^2d\theta$. $\therefore$ area $POQ=\frac{1}{2}r^2d\theta+I$.

Hence the area between the curve and two radii vectores is

$$\tfrac{1}{2}\int_\alpha^\beta r^2\,d\theta.$$

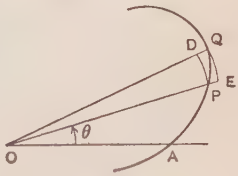

Fig. 97*a*.

(2) The area bounded by two radii vectores and two given curves $r_1=f_1(\theta)$ and $r_2=f_2(\theta)$ is

$$\tfrac{1}{2}\int_\alpha^\beta(r_2^2-r_1^2)\,d\theta \quad \text{or} \quad \tfrac{1}{2}\int_\alpha^\beta(r_2^2+r_1^2)\,d\theta$$

according as the curves lie on the same side or on opposite sides of the origin.

(3) The length $s = \int ds = \int \sqrt{r^2 d\theta^2 + dr^2}$,

taken between assigned limits.

(4) The area of the surface formed by the revolution of the curve about the initial line is (§ 125 (4))

$$2\pi \int r \sin\theta \, ds.$$

EXAMPLES.

1. The cardioid $r = a\cos^2\frac{1}{2}\theta$ (Fig. 89).

(1) The area $= \frac{1}{2}a^2 \int_{-\pi}^{\pi} \cos^4 \frac{1}{2}\theta \, d\theta = \frac{3}{8}\pi a^2$.

(2) The length-element $ds = \sqrt{r^2 d\theta^2 + dr^2} = a \cos \frac{1}{2}\theta \, d\theta$, which does not change sign while θ increases from $-\pi$ to π, hence the whole length of the curve is

$$a \int_{-\pi}^{\pi} \cos \tfrac{1}{2}\theta \, d\theta = 4a.*$$

(3) The surface of revolution about the initial line

$$= 2\pi \int_0^{\pi} r \sin\theta \, ds = 2\pi \int_0^{\pi} a \cos^2\tfrac{1}{2}\theta \,.\, \sin\theta \,.\, a \cos\tfrac{1}{2}\theta \, d\theta$$

$$= 4\pi a^2 \int_0^{\pi} \cos^4\tfrac{1}{2}\theta \sin\tfrac{1}{2}\theta \, d\theta = \tfrac{8}{5}\pi a^2.$$

(The volume $= \frac{1}{3}\pi a^3$, § 178, Ex. 6.)

2. The spiral of Archimedes $r = a\theta$ (Fig. 84).

(1) Let it be required to find the area included between the nth and $(n+1)$th spires. On the former $r = a[2(n-1)\pi + \theta]$, and on the latter $r = a[2n\pi + \theta]$, hence the area between them

$$= \tfrac{1}{2}a^2 \int_0^{2\pi} \Big[(2n\pi + \theta)^2 - \Big(2(n-1)\pi + \theta\Big)^2 \Big] d\theta = 8\pi^3 a^2 n,$$

and is $\therefore$ proportional to n.

---

* A change in the sign of the length-element indicates a cusp, which occurs in this case when $\theta = \pi$. As θ increases the area-element $\frac{1}{2}r^2 d\theta$ can change sign only with r^2, *i.e.*, when r becomes imaginary. Hence if we had integrated between the limits 0 and 2π we should have obtained 0 for the length, whereas the area would have been the same as above.

(2) Show that the area of the first spire (θ varying from 0 to 2π) $= 8\pi^3 a^2/6$.

(3) The length of the curve from the origin to $r = r_1$ is

$$\frac{1}{a}\int_0^{r_1} \sqrt{a^2 + r^2}\, dr \quad \text{(see § 112, Ex. 2)}.$$

This is easily shown to be the same as the length of the parabola $y^2 = 2ax$ from the vertex to $y = r_1$.

3. The lemniscate $r^2 = a^2 \cos 2\theta$ (Fig. 27).

(1) The area $= a^2$.

(2) Show that $rds = a^2 d\theta$.

(3) The surface of revolution about the axis $= 2\pi a^2(2 - \sqrt{2})$.

(4) The surface of revolution about a tangent at the centre $= 4\pi a^2$.

[This tangent being taken as initial line, the equation becomes $r^2 = a^2 \sin 2\theta$.]

4. Prove that the length of any curve $= \int \frac{r\, dr}{\sqrt{r^2 - p^2}}$, and that the area $= \frac{1}{2}\int p\, ds = \frac{1}{2}\int \frac{pr\, dr}{\sqrt{r^2 - p^2}}$.

5. To find the length and area of the logarithmic spiral $r = a^\theta$ (Fig. 87).

(1) Let ψ be the constant angle between the radius vector and the tangent. Then $ds = dr/\cos\psi$, whence

$$s = \int_{r_1}^{r_2} \frac{dr}{\cos\psi} = \frac{r_2 - r_1}{\cos\psi},$$

where r_1 and r_2 are the radii vectores of the extremities of the arc.

(2) For the area, $\frac{1}{2}p\, ds = \frac{1}{2}r \sin\psi\, dr/\cos\psi$.

$$\therefore \text{ area} = \tfrac{1}{2}\tan\psi \int_{r_1}^{r_2} r\, dr = \tfrac{1}{4}(r_2^2 - r_1^2)\tan\psi.$$

6. The length of the spiral $r = e^{-\theta}$ from $\theta = 0$ to $\theta = \infty$ is $\sqrt{2}$.

7. In the curve $r^2 = a^2 \sin 4\theta$ (Fig. 96) show that the area of each loop $= \frac{1}{4}a^2$.

8. In the curve $r = a \sin 4\theta$ (Fig. 95) show that the area of each loop $= \frac{1}{16}\pi a^2 = \frac{1}{2}$ that of the circumscribed circular sector (centre the origin).

9. Prove that the length of the curve $r^{\frac{1}{n}} = a^{\frac{1}{n}} \cos \frac{1}{n}\theta$ is

$$\frac{n(n-2)\ldots}{(n-1)(n-3)\ldots} \cdot 2a\alpha,$$

where α is 1 or $\frac{1}{2}\pi$ according as n is even or odd.

10. In the spiral $r\theta = a$ (Fig. 85) show that the area bounded by two radii vectores and the curve is $\frac{1}{2}a(r_2 - r_1)$.

11. The polar equation of the cissoid (Fig. 41) is $r \cos \theta = a \sin^2\theta$, that of its asymptote is $r \cos \theta = a$, that of the circle of diameter a is $r = a \cos \theta$; show that the area between the cissoid and its asymptote $= \frac{3}{4}\pi a^2$, and that the area between the cissoid and the circle $= (\frac{1}{2}\pi - 1)a^2$.

12. Find the area of a sector of the rectangular hyperbola $r^2 \cos 2\theta = a^2$ (Fig. 98) between $\theta = 0$ and $0 = \alpha$.

Ans. $\frac{1}{4}a^2 \log \tan (\frac{1}{4}\pi + \alpha)$.

13. Find the area of a sector of any hyperbola between $\theta = 0$ and $\theta = \alpha$, the centre being the origin and the transverse axis the initial line.

Ans. $\frac{1}{4}ab \log \left(\frac{b + a \tan \alpha}{b - a \tan \alpha}\right)$.

14. Find the area of an elliptic sector between $\theta = 0$ and $\theta = \alpha$, the centre being the origin and the major axis the initial line.

Ans. $\frac{1}{2}ab \tan^{-1}\left(\frac{a}{b} \tan \alpha\right)$.

15. Show that the area of the limaçon $r = n(1 + e \cos \theta)$ is $\pi n^2(1 + \frac{1}{2}e^2)$.

16. The chord which is drawn through the origin so as to cut off from a given curve a segment of maximum or minimum area is bisected by the origin.

For d (area) $= \frac{1}{2}r_1^2\, d\theta - \frac{1}{2}r_2^2\, d\theta = 0$, $\therefore\ r_1 = r_2$.

17. Find the area enclosed by the curves

(1) $r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$. *Ans.* $\frac{1}{2}\pi(a^2 + b^2)$.

(2) $r^2 = a^2 \cos^2\theta - b^2 \sin^2\theta$. $ab + (a^2 - b^2) \tan^{-1}(a/b)$.

18. The area of the common parabola $r \cos^2 \frac{1}{2}\theta = a$ from $\theta = 0$ to $\theta = \alpha$ is $a^2(\tan \frac{1}{2}\alpha + \frac{1}{3} \tan^3 \frac{1}{2}\alpha)$.

19. If the conchoid $r = a \sec\theta - b$ has a loop, show that the area of the loop is $a\sqrt{b^2-a^2} + b^2\cos^{-1}(a/b) - 2ab\cosh^{-1}(b/a)$.

148. It is in general impossible to obtain the area exactly unless one coördinate can be expressed in terms of the other, or each in terms of a third variable.

When the rectilinear equation of a curve consists of terms of two dimensions only, both x and y are expressible in terms of m, the slope of the line drawn from the origin to (x, y). We can sometimes obtain the area by taking m as the variable.

If $m = \tan\theta$, $dm = \sec^2\theta\, d\theta$, $\therefore \frac{1}{2}r^2\, d\theta = \frac{1}{2}r^2\cos^2\theta\, dm = \frac{1}{2}x^2\, dm$.

$$\therefore \text{ the area} = \tfrac{1}{2}\int x^2\, dm.$$

The area included between two curves will be

$$\tfrac{1}{2}\int (x_2^2 \pm x_1^2)\, dm.$$

EXAMPLES.

1. The ellipse $ax^2 + bxy + cy^2 = k$.
Substituting mx for y, we have $x^2 = k/(a + bm + cm^2)$.
Hence the whole area

$$= \int_{-\infty}^{\infty} \frac{k\, dm}{a + bm + cm^2} = \frac{2\pi k}{\sqrt{4ac - b^2}} \quad \text{(see § 108 (1))}.$$

2. (1) The folium $x^3 + y^3 = 3axy$ (Fig. 28).
Here $x = 3am/(1 + m^3)$, $\therefore$ area of the loop

$$= \tfrac{1}{2}\int_0^{\infty} \frac{9a^2m^2\, dm}{(1 + m^3)^2} = \tfrac{3}{2}a^2 = \tfrac{3}{2}OBAC.$$

(2) On the asymptote $x + y + a = 0$, $m = -a/(1 + m)$; hence the area in the second and fourth quadrants between the curve and the asymptote

$$= \tfrac{1}{2}\int_{-\infty}^{\infty} \left[\frac{a^2}{(1+m)^2} - \frac{9a^2m^2}{(1+m^3)^2}\right] dm = a^2.$$

Adding $\frac{1}{2}a^2$, the area of the triangle ODE, we have the whole

area between the curve and the asymptote $= \frac{3}{2}a^2 =$ the area of the loop.

3. Find the area of the closed part of the curve $a^2y^2(y-x)+x^5=0$.

Ans. $\frac{1}{12}a^2$.

4. Find the area of a loop of the following curves:

(1) $ay^3-3ax^2y=x^4$, Fig. 36. *Ans.* $\frac{3}{8}\frac{6}{5}\sqrt{3}a^2$.

(2) $ay^4-ax^2y^2=x^5$, Fig. 37. $\frac{4}{3}\frac{}{1}\frac{}{5}a^2$.

(3) $x^4+y^4=4a^2xy$. $\frac{1}{2}\pi a^2$.

(4) $ax^2+y^3=axy$. $\frac{1}{6}\frac{}{0}a^2$.

CHAPTER XXXII.

ASSOCIATED CURVES.

Inverse Curves.

149. If on the radius vector r of a curve, a distance r' is measured from the origin so that $rr' = k^2$, where k is constant, the locus of the extremity of r' is called an *inverse* of the given curve. The radius vector of the inverse curve is proportional to the reciprocal of that of the given curve, and its polar equation may be found from that of the given curve by substituting k^2/r for r. Thus (see § 139) the inverse of the equilateral hyperbola with reference to the centre is a lemniscate (Fig. 98), that of a conic section with reference to a focus is a limaçon of the form Figs. 88, 89, or 90, according as e is $<$, $=$, or >1, i.e., according as the conic is an ellipse, parabola, or hyperbola.

Examples.

1. Show that the inverse of a circle with reference to a point on the circumference is a straight line, and that with reference to any other point it is a circle.

2. The angle between the radius vector and the tangent at any point of the inverse is the supplement of the corresponding angle in the given curve.

For, if OPQ, $OP'Q'$ (Fig. 98) are consecutive (see § 84) radii vectores meeting one curve in P, P', and the other in Q, Q', the rectangles $OP \cdot OQ$, $OP' \cdot OQ'$ are equal, $\therefore$ a circle may be described through P, Q, P', Q', $\therefore Q'P'P + PQQ' =$ two right angles; hence, supposing P' to approach P, the tangents at corresponding points

P and Q make supplementary angles with the common radius vector.

Otherwise thus: $r=k^2/r'$, $\therefore \log r=\log k^2-\log r'$,

$$\therefore \frac{1}{r}dr/d\theta=-\frac{1}{r'}dr'/d\theta, \quad \text{or} \quad \cot\psi=-\cot\psi', \quad \therefore \psi'=\pi-\psi.$$

3. Hence show that the inverse of a logarithmic spiral with reference to its origin is a logarithmic spiral.

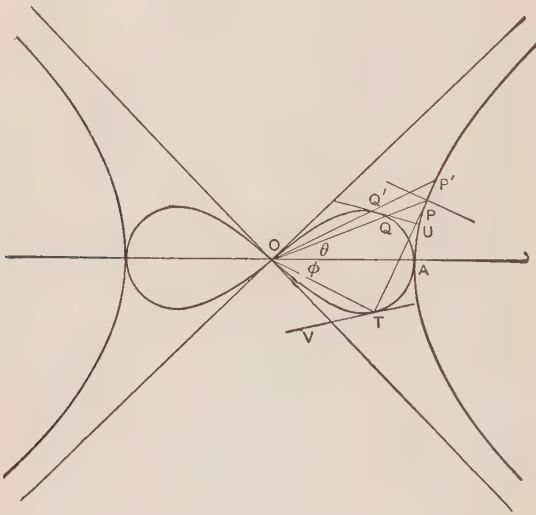

Fig. 98.

4. Show that the curves Figs. 84 and 85, 93 and 95, and 94 and 96 are the inverse of each other.

Pedal Curves.

150. The locus of the foot of the perpendicular from a given point on the tangent to a given curve is called the *pedal* of the curve with reference to the point. For example, the pedal of a parabola with reference to its focus is a straight line (the tangent at the vertex); the pedal of an ellipse or hyperbola with reference to a focus is a circle (the auxiliary circle).

In Fig. 97, T and T' are consecutive points on the pedal, corresponding to P and Q on the given curve; and the limit of position of TT' produced is the tangent to the pedal. If the angle $AOT = \phi$ and $OT = p$, then (ϕ, p) are the polar coördinates of the point on the pedal corresponding to (θ, r) on the given curve. If then we can express p in terms of r, and ϕ in terms of θ, the polar equation of the pedal will be easily obtained from that of the given curve.

EXAMPLES.

1. The pedal of an equilateral hyperbola is a lemniscate (Fig. 98).

For $pr = a^2$ (Ex. 6, § 141), and $\phi = \theta$ (Ex. 2, § 141), hence substituting in $r^2 \cos 2\theta = a^2$ we have $a^2 \cos 2\phi = p^2$, or writing θ and r for ϕ and p, $r^2 = a^2 \cos 2\theta$, the equation of the lemniscate.

In a similar way it may be shown that the pedal of any curve of the form $r^m \cos m\theta = a^m$ is $r^n \cos n\theta = a^n$, where $n \equiv m/(1-m)$, and that the pedal of $r^m = a^m \cos m\theta$ is $r^n = a^n \cos n\theta$, where $n \equiv m/(1+m)$.

2. The angle between the radius vector and tangent at any point of the pedal = that between the radius vector and tangent at the corresponding point of the given curve.

For, in Fig. 97, let TP produced meet $T'Q$ in S. Then O, T, T', S are on the circumference of a circle since the angles at T, T' are right angles, $\therefore OT'T = OST$, and the limits of these angles are the angles referred to in the enunciation. (In Fig. 98, $OPT = OTV$, if PT and TV are tangents.)

3. Prove that the pedal of a circle with reference to any point is a limaçon of the form Figs. 88, 89, or 90, according as the point is inside, on, or outside the circumference. (These figures are the pedals with reference to O of the circles with centres B and radii BA.)

4. Show that the pedal of a logarithmic spiral with reference to its origin is also a logarithmic spiral.

5. Find the pedal of a parabola with reference to its vertex.

Ans. $r \cos \theta = a \sin^2\theta$, the polar equation of the cissoid, Fig. 41. (The directrix of the parabola is the asymptote of the cissoid.)

6. Show that the pedal of the involute of a circle is a spiral of Archimedes. (It will be found that $\tan \psi$ is proportional to the radius vector.)

7. Find the pedal of the ellipse with reference to the centre.

Ans. $r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$.

Polar Reciprocals.

151. The inverse of the pedal of a curve (both pedal and inverse being taken with reference to the same point) is called the *polar reciprocal* of the given curve.

Examples.

1. Show that the polar reciprocal of a circle with reference to any point is a conic section.

2. Find the polar reciprocal of a parabola with reference to its vertex and with reference to its focus, of an ellipse with reference to its centre and with reference to its focus.

3. Show that the polar reciprocal of a logarithmic spiral, $r = a^\theta$, with reference to its origin is another logarithmic spiral.

Roulettes.

152. When one curve rolls on another, the curve described by any point connected with the rolling curve is called a *roulette*.

The simplest case is the *cycloid*, the properties of which have already been considered. Any involute of a curve may also be regarded as the roulette traced by a point in the tangent of the curve as it rolls round the curve.

153. The property of the normal of the cycloid holds for all roulettes, viz., the normal to the roulette at the tracing point passes through the point of contact of the fixed and moving curves, since at each instant the point of contact may be regarded as an instantaneous centre of rotation.

154. When a circle rolls on a straight line any point not on the circumference describes a curve called a *trochoid*, the equations of which are easily shown to be

$$x = a\theta - b \sin\theta,\ y = a - b \cos\theta,$$

where a is the radius of the circle and b the distance of the tracing point from the centre (axes as in Fig. 19).

155. When the circle rolls on the circumference of a fixed circle, the curve described by a point in its circumference is called an *epicycloid* or a *hypocycloid* according as the circle rolls on the outside or inside of the fixed circle. Corresponding to these curves we have *epitrochoids* and *hypotrochoids* described by points not in the circumference.

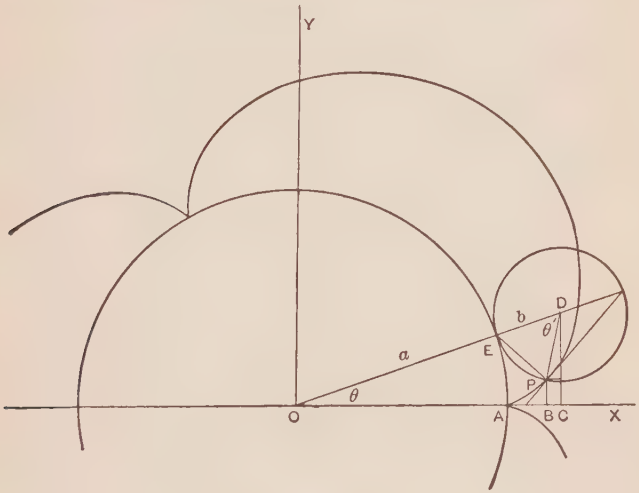

Fig. 99.

For the coördinates of any point P (Fig. 99) on the epicycloid we have

$$x = OB = OD \cos\theta - PD \cos(\theta + \theta').$$

Hence, since arc $PE = b\theta' = AE = a\theta$,

$$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\right)\theta.$$

Similarly, $$y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta.$$

The x and y of a point on the hypocycloid may be obtained in a similar way (or from the epicycloid by changing the sign of b), and are

$$x=(a-b)\cos\theta+b\cos\left(\frac{a-b}{b}\right)\theta.$$

$$y=(a-b)\sin\theta-b\sin\left(\frac{a-b}{b}\right)\theta.$$

The equations of the epitrochoid and hypotrochoid are of the same form, the coefficient b in the second term being changed into h, where h is the distance of the tracing point from the centre of the rolling circle.

EXAMPLES.

1. Show that in any epicycloid

$$ds=2(a+b)\sin\frac{a\theta}{2b}d\theta=2(a+b)\frac{b}{a}\sin\frac{\theta'}{2}d\theta',$$

and hence that the length of the curve from cusp to cusp is $8(a+b)b/a$.

2. Show that the epicycloid is a cardioid when $b=a$.

3. Show that the hypocycloid is the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ (Fig. 18) when $b=\frac{1}{4}a$.

4. When a circle rolls inside another circle of double its diameter, show that every point in the circumference describes a straight line and every other point an ellipse.

5. The radius of curvature at any point of an epicycloid

$$=\frac{4(a+b)b}{a+2b}\sin\frac{a\theta}{2b}=\frac{4(a+b)b}{a+2b}\sin\frac{\theta'}{2}=\frac{2(a+b)}{a+2b}\times\text{chord }EP$$

and is therefore proportional to the chord EP.

For, if the tangent at P make an angle ϕ with OX, $\phi=\theta+\frac{1}{2}\theta'$ and $R=ds/d\phi$ (§ 85).

6. Show in a similar way that in the cycloid

$$x=a(\theta-\sin\theta),\quad y=a(1-\cos\theta),\quad \text{Fig. 60.}$$

$ds=PB\,d\theta$, $\phi=\frac{1}{2}\pi-\frac{1}{2}\theta$, and hence that $R=2PB$.

Envelopes.

156. Let $f(x, y, \alpha)=0$ represent the equation of a curve (i.e. of any plane locus, including a straight line), α being a quantity involved in the equation, but independent of x and y for its value. As α may have any value the equation may be regarded as representing a family of curves. Supposing α to have a certain value in one instance, let it receive an increment $\Delta\alpha$. The two equations

$$f(x, y, \alpha)=0 \quad (1), \qquad f(x, y, \alpha+\Delta\alpha)=0 \quad (2)$$

then represent two curves of the family. Their points of intersection approach limits of position as $\Delta\alpha \doteq 0$. The locus of these point-limits for all values of α is called the envelope of the family of curves.

The quantity α is called a variable parameter.

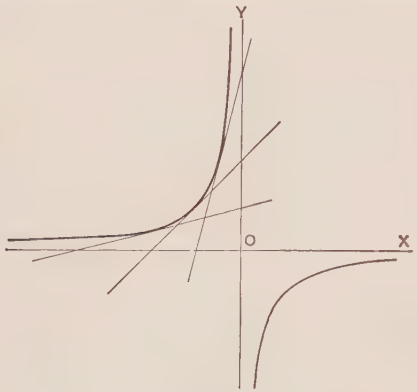

FIG. 100.

Ex. $y=\alpha^2x+\alpha$ represents a family of straight lines. Consider the two for which α is 1 and $1+\Delta\alpha$ respectively. The lines $y=x+1$, $y=(1+\Delta\alpha)^2x+(1+\Delta\alpha)$ intersect in the point $\left(-\frac{1}{2+\Delta\alpha}, \frac{1+\Delta\alpha}{2+\Delta\alpha}\right)$. The limit of this when $\Delta\alpha \doteq 0$ is $(-\frac{1}{2}, \frac{1}{2})$. This is therefore one point on the envelope of the family.

157. Equation of the envelope. The points of intersection of (1) and (2), § 156, lie on the curve

$$\frac{f(x, y, \alpha + \Delta\alpha) - f(x, y, \alpha)}{\Delta\alpha} = 0, \tag{3}$$

since this equation is satisfied by any simultaneous values of x and y which make $f(x, y, \alpha)$ and $f(x, y, \alpha + \Delta\alpha)$ separately $= 0$. As $\Delta\alpha \doteq 0$ the limit of (3) is

$$\frac{\partial f(x, y, \alpha)}{\partial \alpha} = 0, \tag{4}$$

the differentiation being partial since only α varies. The point-limits of the intersections of (1) and (2) therefore lie on (4), and their locus, the envelope, is obtained by eliminating (α) from (1) and (4).

Ex. Equation (4) for $y = \alpha^2 x + \alpha$ is $2\alpha x + 1 = 0$. Eliminating α, $4xy = -1$ The envelope is therefore a rectangular hyperbola (Fig. 100).

158. Prop. *The envelope touches every curve of the family.* Let u stand for $f(x, y, \alpha)$, and suppose (x, y) to be a point common to the envelope and curve (1), § 156. For dy/dx, the slope of the tangent of (1), we have (§ 47),

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0,$$

α being constant. We may consider (1) to be also the equation of the envelope, α being a variable, viz., that function of x and y obtained from (4). Hence for the envelope

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial \alpha} d\alpha = 0.$$

But $\partial u/\partial \alpha = 0$ from (4). Hence dy/dx at (x, y) is the same for (1) and the envelope.

Ex. In the example of §§ 156, 157, $y = x + 1$ and the envelope $4xy = -1$ touch at the point $(-\frac{1}{2}, \frac{1}{2})$.

159. The given equation may contain two or more variable parameters, subject however to other relations connecting them, whereby all except one may be eliminated from the given equation.

Ex. To show that all ellipses having the same centre and area, and their axes in the same directions, touch a pair of hyperbolas of which the axes are asymptotes.

We have to find the envelope of $x^2/\alpha^2+y^2/\beta^2=1$, where $\alpha\beta=k^2$, a constant, whence $\beta=k^2/\alpha$.

Substituting, the equation becomes $x^2/\alpha^2+\alpha^2y^2/k^4=1$.

Differentiating with regard to α, $-2x^2/\alpha^3+2\alpha y^2/k^4=0$.

Eliminating α, $xy=\pm\frac{1}{2}k^2$, the envelope (Fig. 101).

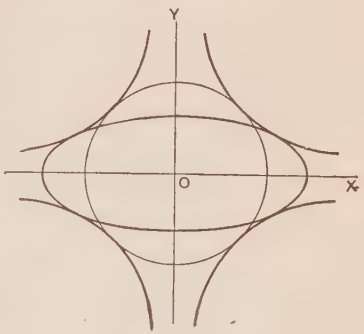

Fig. 101.

Examples.

1. Two sides of a right-angled triangle are given in position and the area is constant, find the envelope of the hypotenuse.

Ans. A rectangular hyperbola.

2. Particles are projected in the same vertical plane with the same velocity v, but at different elevations; show that their paths all touch the parabola

$$x^2=-\frac{2v^2}{g}\left(y-\frac{v^2}{2g}\right),$$

of which the point of projection is the focus.

[In other words find the envelope of

$$y=x\tan\alpha-gx^2/(2v^2\cos^2\alpha).]$$

3. Show that the circles described on the double ordinates of the parabola $y^2=4ax$ as diameters touch the equal parabola $y^2=4a(x+a)$.

4. Find the envelope of $u\alpha^2+v\alpha+w=0$, where u, v, w are functions of x and y. *Ans.* $v^2=4uw$.

The result is the same as the condition that the given equation should have equal roots. Explain.

5. Find the envelopes of

$$u\cos^m\theta + v\sin^m\theta = w, \tag{1}$$

$$u\sec^m\theta - v\tan^m\theta = w. \tag{2}$$

Ans. (1) $u^n + v^n = w^n$ } where $n \equiv \frac{2}{2-m}$.
(2) $u^n - v^n = w^n$ }

Many examples may be reduced to these by observing that a condition of the form $\left(\frac{\alpha}{a}\right)^r + \left(\frac{\beta}{b}\right)^r = 1$ is equivalent to the two relations $\alpha = a\cos^{\frac{2}{r}}\theta$, $\beta = b\sin^{\frac{2}{r}}\theta$, while $\left(\frac{\alpha}{a}\right)^r - \left(\frac{\beta}{b}\right)^r = 1$ is equivalent to $\alpha = a\sec^{\frac{2}{r}}\theta$, $\beta = b\tan^{\frac{2}{r}}\theta$.

6. Find the envelope of a line which moves in such a way that the sum of its intercepts on the axes is constant.

We have $\frac{x}{\alpha} + \frac{y}{\beta} = 1$, and $\alpha + \beta = k$. We may substitute the value of β and then differentiate, or we may proceed as follows:

Let $\alpha = k\cos^2\theta$, $\beta = k\sin^2\theta$; the line becomes $x(\cos\theta)^{-2} + y(\sin\theta)^{-2} = k$, hence (Ex. 5) the envelope is $x^{\frac{1}{2}} + y^{\frac{1}{2}} = k^{\frac{1}{2}}$, a parabola touching the axes.

7. A straight line of given length k moves with its extremities on two rectangular axes, find the envelope of the line.

Ans. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = k^{\frac{2}{3}}$, a four-cusped hypocycloid.

8. Given in position the axes of an ellipse and that their sum $= 2k$, show that the ellipse touches the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = k^{\frac{1}{2}}$.

9. From any point in $\left(\frac{x}{a}\right)^m \pm \left(\frac{y}{b}\right)^m = 1$ perpendiculars are drawn to meet the axes in A and B, find the envelope of AB.

Ans. $\left(\frac{x}{a}\right)^n \pm \left(\frac{y}{b}\right)^n = 1$, where $n \equiv \frac{m}{m+1}$.

10. To the ellipse or hyperbola $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ pairs of tangents are drawn from points in the ellipse $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$, show that the chords

of contact touch the ellipse

$$\left(\frac{a_1x}{a^2}\right)^2+\left(\frac{b_1y}{b^2}\right)^2=1.$$

11. When the tangents are drawn from points in the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, show that the chords of contact touch the hyperbola

$$\left(\frac{a_1x}{a^2}\right)^2-\left(\frac{b_1y}{b^2}\right)^2=1.$$

12. The evolute of a curve may be considered to be the envelope of its normals; find in this way the evolute of an ellipse.

The normal at (α, β) is $\frac{a^2x}{\alpha}-\frac{b^2y}{\beta}=a^2-b^2$,

or, writing $a\cos\theta$ for α, and $b\sin\theta$ for β,

$$x\,.\,a(\cos\theta)^{-1}-y\,.\,b\,(\sin\theta)^{-1}=a^2-b^2,$$

the envelope of which is (Ex. 5),

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}},$$

which is therefore the evolute (cf. § 89).

13. Show in a similar way that the evolute of the hyperbola is

$$(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}=(a^2+b^2)^{\frac{2}{3}}.$$

14. Parallel rays of light are reflected from the circumference of a circle. Find the envelope of the reflected rays.

Take the centre for origin and the x-axis parallel to the incident rays. The equation of the ray reflected from the point $(a\cos\theta, a\sin\theta)$ is

$$x\sin 2\theta-y\cos 2\theta-a\sin\theta=0,$$

whence the envelope is

$$x=\tfrac{1}{4}a(3\cos\theta-\cos 3\theta),\quad y=\tfrac{1}{4}a(3\sin\theta-\sin 3\theta),$$

an epicycloid formed by a circle of radius $\frac{1}{4}a$ on a circle of radius $\frac{1}{2}a$

CHAPTER XXXIII.

CENTRES OF GRAVITY.

160. In finding the coördinates $\bar{x}$, $\bar{y}$ of the centre of gravity of a body, we suppose the body to be divided into parts of weights $w_1, w_2 \ldots$, and of which the centres of gravity are the points (x_1, y_1), $(x_2, y_2), \ldots$, then equate the sum of the moments of the weights to the moment of the sum of the weights if placed at the centre of gravity. Thus supposing gravity perpendicular to the x-axis we have

$$w_1x_1+w_2x_2+\ldots=(w_1+w_2+\ldots)\bar{x},$$

$$\therefore\ \bar{x}=\frac{w_1x_1+w_2x_2+\ldots}{w_1+w_2+\ldots}\equiv\frac{\Sigma wx}{\Sigma w}.$$

Similarly supposing the body and the axes placed so that gravity is perpendicular to the y-axis, we have

$$\bar{y}=\frac{w_1y_1+w_2y_2+\ldots}{w_1+w_2+\ldots}\equiv\frac{\Sigma wy}{\Sigma w}.$$

These formulæ also hold when the points are not in one plane, there being also a third coördinate,

$$\bar{z}=\Sigma wz/\Sigma w.$$

If the parts referred to are infinitesimal the sign of integration replaces that of summation to indicate the limit of a sum.

161. The division into parts and the limits of the summation in the following cases are the same as if we were about to calculate an area, volume, or length. The bodies are assumed to be homogeneous (of uniform density) and

hence weight is proportional to volume. For such bodies the centre of gravity is also known as the *centroid*.

162. An area. To find the c.g. of a thin plate or lamina * of the form $ABDC$, Fig. 61, we have the element of area $= y\,dx$; element of weight $= w\,.\,y\,dx$, where $w =$ weight per unit area; c.g. of element at $(x+i,\ \frac{1}{2}y)$, where i is infinitesimal; hence element of moment $= wy\,dx\,.\,x$ when gravity is perpendicular to the x-axis, and $= wy\,dx\,.\,\frac{1}{2}y$ when gravity is perpendicular to the y-axis. Dividing the sum-limit of the moments by that of the weights (§ 160),

$$\bar{x} = \frac{\int xy\,dx}{\int y\,dx},\quad \bar{y} = \tfrac{1}{2}\frac{\int y^2\,dx}{\int y\,dx},$$

when w (which is assumed to be constant) is cancelled.

It will be noticed that the denominator = the area.

163. A solid of revolution about OX. Element of volume $= \pi y^2\,dx$, of weight $= w\,.\,\pi y^2\,dx$, w being the weight per unit volume, element of moment $= w\,.\,\pi y^2\,dx\,.\,x$,

$$\therefore\ \bar{x} = \frac{\int xy^2\,dx}{\int y^2\,dx},\quad \bar{y} = 0.$$

The denominator = volume/π.

164. An arc. Proceeding as above we have for the c.g. of a material line in the form of the curve CD,†

$$\bar{x} = \frac{\int x\,ds}{\int ds},\quad \bar{y} = \frac{\int y\,ds}{\int ds}.$$

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* Results for a lamina are limits for a uniform and infinitesimal thickness.

† The results are limits for a body of uniform and infinitesimal cross-section.

165. A surface of revolution about OX. For a curved surface of this form and of infinitesimal thickness,

$$\bar{x} = \frac{\int xy\, ds}{\int y\, ds}, \quad \bar{y} = 0.$$

166. An area in polar coördinates. Element of area $= \frac{1}{2}r^2\, d\theta$ (§ 147); its c.g. is distant $\frac{2}{3}r + i$ from the origin; * hence if the initial line is taken as x-axis,

$$\therefore\ \bar{x} = \frac{\int w \,.\, \frac{1}{2}r^2\, d\theta \,.\, \frac{2}{3}r\cos\theta}{\int w \,.\, \frac{1}{2}r^2\, d\theta} = \frac{2}{3}\frac{\int r^3 \cos\theta\, d\theta}{\int r^2\, d\theta}$$

and

$$\bar{y} = \frac{2}{3}\frac{\int r^3 \sin\theta\, d\theta}{\int r^2\, d\theta}.$$

167. The subject may also be considered from the point of view of geometry only. Let a be an area-element or volume-element which is infinitesimal in every direction, and which contains a point (x, y, z). Then the limits of

$$\frac{\Sigma ax}{\Sigma a}, \quad \frac{\Sigma ay}{\Sigma a}, \quad \frac{\Sigma az}{\Sigma a}$$

are the same as the $\bar{x}, \bar{y}, \bar{z}$ of § 160 for homogeneous bodies, and are the coördinates of a point which is called the centroid (or centre of gravity) of the area or volume. Or, a may be taken as a mass-element, in which case the point is called the centre of mass (or centre of gravity) of the body.

168. Pappus's (or Guldin's) properties of the centre of gravity. From § 164 we have $\int y\, ds = \bar{y}\int ds$, and multiplying both sides by 2π,

* The c.g. of a triangle is assumed to be the point of intersection of the medians.

$$\int 2\pi y \, . \, ds = \left(\int ds\right) \, . \, 2\pi\bar{y}. \qquad (1)$$

Similarly from § 162,

$$\int \pi y^2 \, dx = \left(\int y \, dx\right) \, . \, 2\pi\bar{y}. \qquad (2)$$

These results are equivalent to the following statements, which are known as Pappus's or Guldin's Properties:

(1) The surface of a solid of revolution is equal to the length of the revolving curve multiplied by the length of the path of the c.g. of the curve (i.e. *of the arc*).

(2) The volume of a solid of revolution is equal to the revolving area multiplied by the length of the path of the c.g. of this area.

N.B. The axis of revolution may touch but not cut the curve.

Examples.

1. The parabolic area OAB, Fig. 74. *Ans.* $\bar{x} = \frac{3}{5}x_1$, $\bar{y} = \frac{3}{8}y_1$. Of the solid of revolution round OX, $\bar{x} = \frac{3}{4}x_1$.

2. The quadrant of an ellipse. *Ans.* $\bar{x} = \dfrac{4a}{3\pi}$, $\bar{y} = \dfrac{4b}{3\pi}$

3. Half of a prolate spheroid. *Ans.* $\bar{x} = \frac{3}{8}a$.

4. The circle $x^2 + y^2 = 2ax$ between $x = 0$ and $x = h$ revolves about the axis of x, find the c.g. of the volume of the spherical segment thus formed.

Ans $\bar{x} = \left(\dfrac{8a - 3h}{3a - h}\right)\dfrac{h}{4}$.

For a hemisphere this $= \frac{5}{8}a$.

Show that for the surface of the segment $\bar{x} = \frac{1}{2}h$.

5 A circular arc.

Ans. Distance from centre of circle = chord × radius/arc.

For a quadrant this $= 2\sqrt{2}a/\pi$, and $= 2a/\pi$ for a semicircular arc.

6. A circular sector.

Ans. Distance from centre = $\frac{2}{3}$ chord × radius/arc.

For a quadrant this $= 4\sqrt{2}a/3\pi$ and $= 4a/3\pi$ for a semicircle.

7. A circular segment.

Ans. Distance from centre = chord^3/(12 × area of segment).

8 Surface and volume of a right circular cone.

Ans. Distance from vertex = (1) $\frac{2}{3}$ axis, (2) $\frac{3}{4}$ axis.

9. The area between the curve $y = \sin x$ (Fig. 65) and the axis of x, from $x = 0$ to $x = \pi$. *Ans.* $\bar{x} = \frac{1}{2}\pi$, $\bar{y} = \frac{1}{8}\pi$.

10. The cycloid, Fig. 19. *Ans.* Distance from base = $\frac{5}{6}a$.

11. A quadrant of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (Fig. 18).

Ans. $\bar{x} = 256a/315\pi = \bar{y}$.

Of the arc, $\bar{x} = \frac{8}{5}a = \bar{y}$.

12. A quadrant of the whole area of the curve $a^2y^2 = x^2(a^2 - x^2)$, (Fig. 69). *Ans.* $\bar{x} = \frac{3}{16}\pi a$, $\bar{y} = \frac{1}{5}a$.

13. The area between the curve $y^2(a^2 - x^2) = a^4$ and the asymptote $x = a$. *Ans.* $\bar{x} = 2a/\pi$, $\bar{y} = 0$.

14. Find by Pappus's Properties the surface and volume of a torus (or anchor ring), formed by the revolution of a circle of radius a when the centre describes a circle of radius b, $b \geqq a$.

Ans. $4\pi^2ab$, $2\pi^2a^2b$.

15. Find by Pappus's Properties the c.g. of the arc of a semicircle and that of the area of a semicircle.

CHAPTER XXXIV.

MOMENTS OF INERTIA.

169. The Moment of Inertia is a quantity which is often required in connection with the motion of a body about an axis. The following is an illustration.

170. Kinetic energy of rotation. Let it be required to find the kinetic energy which a body possesses on account of its rotation about an axis.

Let the perpendicular distance of a particle of mass m_1 from the axis be r_1 and let ω = the angular velocity of the body about the axis. Then the kinetic energy of the particle

$$= \tfrac{1}{2} \text{ (mass)} \times \text{(linear velocity)}^2 = \tfrac{1}{2} m_1 (\omega r_1)^2 = \tfrac{1}{2} \omega^2 m_1 r_1^2,$$

and the whole kinetic energy of the body,

$$= \tfrac{1}{2} \omega^2 (m_1 r_1^2 + m_2 r_2^2 + \dots) = \tfrac{1}{2} \omega^2 I,$$

where $I \equiv m_1 r_1^2 + m_2 r_2^2 + \dots$

The quantity I is called the moment of inertia of the body about, or with reference to, the axis; hence the following definition:

The *Moment of Inertia* of a body about an axis is the sum of the products obtained by multiplying the mass of each particle of the body by the square of its distance from the axis.

Since the particles of a body are infinitesimal portions of the body, the moment of inertia is obtained as follows: Imagine the body to be divided into parts which are infinitesimal in every direction, and find the limit of the sum of the products of the mass of each part by the square of the distance of some point in it from the given axis.

Since both factors of the product mr^2 are essentially +, the moment of inertia is always +, and the moment of inertia of a body about any axis is always equal to the arithmetical sum of the moments of inertia of its parts about that axis.

171. Prop. The m.i. of a body about any axis = the m.i. about a parallel axis through the centre of gravity $+Mh^2$, where M is the mass of the body and h is the distance between the parallel axes.

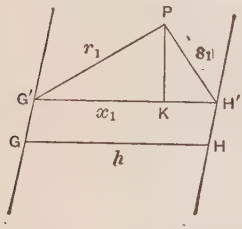

FIG. 102.

Take at a point P in the body a particle of mass m_1. Let a plane through P perpendicular to the axes in question meet the one through the centre of gravity G in G' and the parallel one in H', then $G'H' = h$. Draw PK perpendicular to $G'H'$ and let $G'K = x_1$. Then

$$s_1^2 = r_1^2 + h^2 - 2hx_1 \text{ (Euc. II. 13)},$$

$$\therefore\ m_1 s_1^2 = m_1 r_1^2 + m_1 h^2 - 2h m_1 x_1.$$

Similarly for particles m_2, m_3, etc.

$$\therefore m_1 s_1^2 + m_2 s_2^2 + \ldots = (m_1 r_1^2 + m_2 r_2^2 + \ldots) + h^2(m_1 + m_2 + \ldots) - 2h(m_1 x_1 + m_2 x_2 + \ldots).$$

The left-hand side = the m.i. about the axis through H; and of the three terms on the right, the first = the m.i. about the parallel axis through the centre of gravity, the second =

Mh^2, and the third $=0$ (§ 160), since the centre of gravity is in the line from which $x_1, x_2, \ldots$ are measured.

172. The proposition just proved is true for all bodies, but the following applies only to laminæ.

Prop. Let $X'X$, $Y'Y$ be two lines in the plane of a lamina and meeting *at right angles* in O, and let $Z'Z$ be a line through O perpendicular to the plane.

Let I_1 = the m.i. of the lamina about $X'X$, I_2 = that about $Y'Y$, I = that about $Z'Z$; then

$$I = I_1 + I_2.$$

For $r_1^2 = x_1^2 + y_1^2$,

$$\therefore\ m_1 r_1^2 = m_1 x_1^2 + m_1 y_1^2.$$

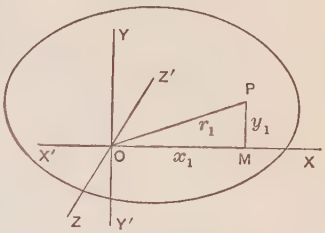

FIG. 103.

The proposition is therefore true for a particle at P, and hence it is true for all the particles of the lamina.

173. When the m.i. is put into the form Mk^2 (M being the mass), k is called the *radius of gyration;* hence the radius of gyration of a body with reference to an axis is the distance from the axis of a point at which a particle having the same mass as that of the body may be placed so that its m.i. may be the same as that of the body.

If k = the radius of gyration with reference to an axis passing through the centre of gravity, and k_1 that about a parallel axis at a distance h, we have $k_1^2 = k^2 + h^2$, since (§ 171)

$$Mk_1^2 = Mk^2 + Mh^2.$$

174. In the following examples the density, i.e., the mass per unit volume, is represented by μ, and the bodies are assumed to be homogeneous, i.e., of uniform density, unless the contrary is specified.

Ex. 1. To find the m.i. of a rectangular lamina whose sides are a, a, b, b, about an axis bisecting the sides a, a, Fig. 104.

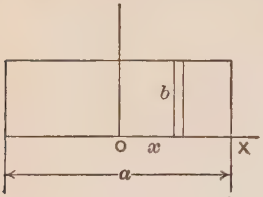

Fig. 104.

Divide the rectangle into parallel strips of length b and width dx, and measure x from the axis. The elements are as follows: area $= b\,dx$, volume $= t\,.\,b\,dx$, where $t =$ the thickness of the lamina, mass $= \mu\,.\,tb\,dx$, m.i. $= \mu tb\,dx\,.\,x^2$, since every particle is at a distance $x+i$ from the axis, i being infinitesimal. Integrating between 0 and $\frac{1}{2}a$ and doubling we have for the m.i. of the whole rectangle

$$2\int_0^{\frac{a}{2}} \mu tbx^2\,dx = \tfrac{1}{12}\mu tba^3 = (\mu tab)\frac{a^2}{12}$$

The quantity in parentheses is the whole mass ($= \mu \times$ volume), $\therefore$ the m.i. $= m\frac{a^2}{12}$, and hence the radius of gyration $= \frac{a}{\sqrt{12}}$. Similarly the m i. about the axis bisecting the sides b, b is $m\frac{b^2}{12}$.

2. The m.i. of the rectangle about a normal axis through the intersection of the two axes of Ex. 1 is § (172) $m\frac{a^2+b^2}{12}$

The same formula is true for any parallelogram (of which a and b are adjacent sides) about an axis drawn as in this case through the intersection of the diagonals at right angles to the plane.

3. The m.i. of the rectangle about a side b is (§ 171),

$$m\frac{a^2}{12} + m\left(\frac{a}{2}\right)^2 = m\frac{a^2}{3}.$$

4. The m.i. of the rectangle about a normal axis through one angle $= m\frac{a^2+b^2}{3}$.

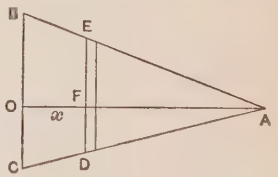

Fig. 105.

5. Any triangular lamina (Fig. 105) about one side BC. Let $BC = a$, the perpendicular $OA = h$. Then $DE : BC :: FA : OA$,

$\therefore\ DE/a=(h-x)/h,\ \therefore\ DE=(h-x)a/h.$

$$\therefore\ \text{m.i.}=\int_0^h \mu(h-x)\frac{a}{h}\,.\,t\,.\,dx\,.\,x^2=m\frac{h^2}{6}.$$

6. A circular lamina of radius r about a normal axis through the centre.

Consider the annulus between the concentric circles of radii x and $x+dx$. The elements are: area $=2\pi x\,.\,dx$, mass $=\mu t\,.\,2\pi x\,dx$, m.i. $=\mu t\,.\,2\pi x\,dx\,.\,x^2$, since every particle of the annulus is at a distance $x+i$ from the axis, i being infinitesimal,

$$\therefore\ \text{whole m.i}=\int_0^r 2\mu\pi t x^3\,dx=\tfrac{1}{2}\mu\pi t r^4=m\frac{r^2}{2}$$

7. A circular lamina about a diameter.

Let the required m.i. $=I$. The sum of the moments of inertia about two diameters at right angles to each other $=2I$; it also (by § 172 and Ex. 6) $=m\frac{r^2}{2}$, $\therefore\ I=m\frac{r^2}{4}$.

8. A circular lamina about a normal axis through the centre when the density is supposed to vary inversely as the distance from the centre.

Let $\mu=k/x$, where k is a constant. Then m.i.

$$=\int_0^r\left(\frac{k}{x}\right)\,.\,2\pi t x^3\,dx=\tfrac{2}{3}\pi t k r^3.$$

But the mass $m=\int_0^r\left(\frac{k}{x}\right)\,.\,2\pi x\,dx\,.\,t=2\pi k t r.$

$$\therefore\ \text{m.i.}=m\frac{r^2}{3}.$$

9. An ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its minor axis.

The m.i. $=4\int_0^a \mu\,.\,y\,dx\,.\,t\,.\,x^2$. Substitute y from the equation of the curve and let $x=a\sin\theta$. The result is $m\frac{a^2}{4}$.

Similarly about the major axis the m.i. is $m\frac{b^2}{4}$.

10. A sphere about a diameter.

Take the diameter as x-axis and the centre as origin.

Consider the sphere to be made up of laminæ perpendicular to the axis, and of thickness dx Then m.i. (see Ex. 6)

$$=2\int_0^r \mu \,.\, \pi y^2 \,.\, dx \,.\, \frac{y^2}{2} \quad \text{and} \quad y^2=r^2-x^2, \quad \text{whence the m.i.} = m\tfrac{2}{5}r^2.$$

11. A right circular cylinder of radius r about its geometrical axis.

The cylinder may be considered as made up of circular laminæ perpendicular to the axis, hence (Ex. 6) the m.i. $=m\frac{r^2}{2}$.

Similarly for a cube, a right prism, etc., about an edge or any parallel axis.

12. A right circular cylinder of radius r and length l about an axis bisecting at right angles the geometrical axis.

As before, suppose the cylinder to be made up of circular laminæ. The mass of the lamina at a distance x from the axis $=\mu \,.\, \pi r^2 \; dx$, and its m.i. about a diameter in its own plane and parallel to the given axis $=$ mass $\times \frac{r^2}{4}$ (Ex. 7), $\therefore$ its m.i. about the given axis $=$ mass $\left(\frac{r^2}{4}+x^2\right)$ (§ 171);

$$\therefore \text{ whole m.i.} = 2\int_0^{\frac{1}{2}l} \mu\pi r^2 \; dx \left(\frac{r^2}{4}+x^2\right) = m\left(\frac{r^2}{4}+\frac{l^2}{12}\right).$$

175. As in Ch. XXXIII, the subject may be considered from the point of view of geometry alone. If a is an area (or volume) element which is infinitesimal in every direction, and r the perpendicular distance of some point in it from a straight line, the limit of Σar^2 is called the moment of inertia of the area (or volume) with reference to the straight line. The results are the same as those calculated for homogeneous bodies, area (or volume) taking the place of mass.

EXAMPLES.

Find the moment of inertia of

1. A triangle about (1) an axis through the centre of gravity parallel to the base, (2) a parallel axis through the vertex.

Ans. (1) $m\frac{h^2}{18}$, (2) $m\frac{h^2}{2}$.

2. A rectangle (a by b) about a diagonal. *Ans.* $m\frac{a^2b^2}{6(a^2+b^2)}$.

3. An isosceles triangle about a normal axis through the middle point of the base.

Ans. $m\frac{4\text{ alt.}^2+\text{base}^2}{24}$.

4. An isosceles triangle about a normal axis through the vertex.

Ans. $m\frac{12\text{ alt.}^2+\text{base}^2}{24}$.

5. A circular annulus of radii R, r about a normal axis through the centre.

Ans. $m\frac{R^2+r^2}{2}$.

6. A circular annulus about a diameter. *Ans.* $m\frac{R^2+r^2}{4}$.

7. A circle about a tangent. *Ans.* $m\frac{5}{4}r^2$.

8. A circular arc of length s, radius r, and chord c, about an axis through its middle point perpendicular to its plane.

Ans. $m\,.\,2r^2\left(1-\frac{c}{s}\right)$

9. A circle about a normal axis through a point in the circumference. *Ans.* $m\frac{3}{2}r^2$.

10. A parabolic area, Fig. 74, about the x-axis. *Ans.* $m\frac{{y_1}^2}{5}$

11. The same about the y-axis. *Ans.* $m\frac{3}{7}{x_1}^2$.

12. A spherical shell of infinitesimal thickness about a diameter. *Ans.* $m\frac{2}{3}r^2$.

13. A right circular cone of radius r and altitude h about its geometrical axis. *Ans.* $m\frac{3}{10}r^2$.

14. The same about an axis through the vertex perpendicular to the geometrical axis. *Ans.* $m\frac{12h^2+3r^2}{20}$

15. The same about an axis through the centre of gravity perpendicular to the geometrical axis.

Ans. $m\dfrac{3h^2+12r^2}{80}$.

16. An oblate spheroid about its geometrical axis.

Ans. $m\frac{2}{5}a^2$

17 Any area $ABDC$ (Fig. 61) about the x-axis.

Ans. $\dfrac{m}{3}\dfrac{\int y^3\,dx}{\int y\,dx}$.

18. The hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ (Fig. 18) about the x-axis.

Ans. $m\frac{7}{64}a^2$

CHAPTER XXXV.

SUCCESSIVE INTEGRATION.

176. Successive integration. Let T be a function of u, v, and w, and suppose the following operation to be performed: $T\,dw$ is integrated between the limits w_1 and w_2, u and v being treated as constants; the result is multiplied by dv and integrated between v_1 and v_2, u being treated as a constant; the result is multiplied by du and integrated between u_1 and u_2. The whole operation is indicated by the notation

$$\int_{u_1}^{u_2}\int_{v_1}^{v_2}\int_{w_1}^{w_2} T\,du\,dv\,dw.$$

The limits of w may be functions of u and v, the limits of v may be functions of u, but the limits of u are constants. Instead of three variables there may be two or four, or more.

Ex. 1. $\displaystyle\int_1^2\int_0^x x^2y\,dx\,dy = \int_1^2\left(\int_0^x x^2y\,dy\right)dx = \int_1^2 \tfrac{1}{2}x^4\,dx = 3\tfrac{1}{10}.$

2. $\displaystyle\int_0^3\int_0^x\int_0^{x+y}(x-y)dx\,dy\,dz = \int_0^3\int_0^x (x^2-y^2)dx\,dy$
$$= \int_0^3 \tfrac{2}{3}x^3\,dx = 13\tfrac{1}{2}.$$

3. $\displaystyle\int_1^2\int_2^3 6x^2y\,dx\,dy = 35$

4. $\displaystyle\int_1^2\int_0^{2x}\int_0^{x+y} dx\,dy\,dz = 9\tfrac{1}{3}.$

5. $\displaystyle\int_0^{\pi}\int_0^{a\theta} r\,d\theta\,dr = \tfrac{1}{6}\pi^3a^2.$

6. $\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^3 \sin\theta\cos\theta\, d\phi\, d\theta\, dr = \frac{4}{3}\pi a^4.$

Applications of Successive Integration.

177. Plane area. Rectangular coördinates. Let P be the point (x, y), and let PR, PS be dx, dy, infinitesimal increments of x and y. The rectangle $PQ = dx\, dy$ may be taken as an element of area. To illustrate the method of finding the limit of the sum of such elements by successive integration, let it be required to find the area of the figure KM, which is bounded by the lines $x=a$, $x=b$, and the two curves KL or $y=f(x)$ and NM or $y=F(x)$.

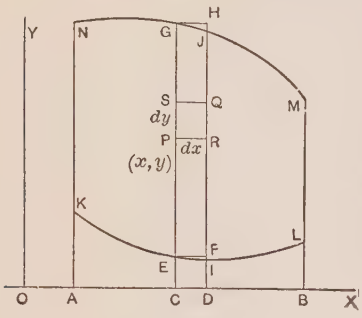

FIG. 106.

$$(1)\quad \left(\int_{CE}^{CG} dy\right) dx = EG\,.\,dx, \qquad (2)\quad \int_a^b EG\,.\,dx = KM.$$

The first result is the sum of such rectangles as PQ, which make up the rectangle EH, which is equivalent (§ 15) to the strip $EIJG$ of the given figure, the second is the sum of such strips for the whole area. The final result is the limit of the sum of such rectangles as PQ when both dx and $dy \doteq 0$.

The whole operation is indicated by the "double integral"

$$\int_a^b \int_{y_1}^{y_2} dx\, dy \quad \text{or} \quad \int_a^b \int_{f(x)}^{F(x)} dx\, dy,$$

y_1 and y_2 being the y's of the curves $y=f(x)$, $y=F(x)$.

More generally, let u be a function of x or y, or of both x and y. The limit of the sum of such products as $u\, dx\, dy$ taken for all parts of the area KM is

$$\int_a^b \int_{y_1}^{y_2} u\, dx\, dy.$$

Ex. 1. Find $\mathcal{L}\Sigma\, y\,dx\,dy$ for a quadrant of the circle $x^2+y^2=a^2$.

We are to obtain $\int_0^a \int_0^{y_1} y\,dx\,dy$, where y_1 is the y of $x^2+y^2=a^2$.

$$\int_0^a \int_0^{y_1} y\,dx\,dy = \int_0^a \left(\int_0^{y_1} y\,dy\right) dx = \int_0^a \tfrac{1}{2}{y_1}^2\,dx = \int_0^a \tfrac{1}{2}(a^2-x^2)dx = \tfrac{1}{3}a^3.$$

This is the moment of the area with reference to the x-axis.

2. Find $\mathcal{L}\Sigma\, y^2\,dx\,dy$ for the same area.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\,dx\,dy = \tfrac{1}{16}\pi a^4.$$

This is the second moment, or moment of inertia, of the area with reference to the x-axis.

3. Find $\mathcal{L}\Sigma\, xy\,dx\,dy$ for the same area. *Ans.* $\frac{1}{8}a^4$.

This is the product of inertia of the area with reference to the axes.

4. Show that the volume of a solid of revolution about the x-axis is

$$2\pi \iint y\,dx\,dy,$$

and deduce the formula of § 95.

178. Plane area. Polar coördinates. Let P be the point $(\theta,\ r)$, and let $POR=d\theta$, $PS=dr$. With centre O describe the arcs PR, SQ. Then the element of area PQ

$$=\tfrac{1}{2}(r+dr)^2\,d\theta-\tfrac{1}{2}r^2\,d\theta$$

$$=r\,d\theta\,dr+\tfrac{1}{2}\,dr^2\,d\theta,$$

the last term being a higher infinitesimal.

$$\therefore\ \mathcal{L}\Sigma\ PQ=\mathcal{L}\Sigma\ r\,d\theta\,dr.$$

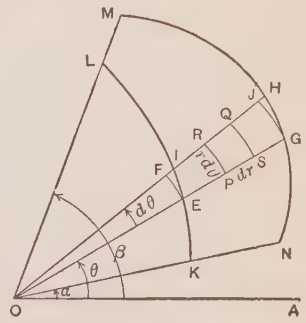

Fig. 107.

Let $KLMN$ be a figure bounded by the lines $\theta=\alpha$, $\theta=\beta$, and the curves KL or $r=f(\theta)$ and NM or

$r=F(\theta)$. Proceeding as in § 177 we find the area

$$=\int_{\alpha}^{\beta}\int_{r_1}^{r_2} r\,d\theta\,dr, \quad \text{or} \quad \int_{\alpha}^{\beta}\int_{f(\theta)}^{F(\theta)} r\,d\theta\,dr,$$

r_1 and r_2 being the r's of the given curves.

The first integral $\frac{1}{2}(r_2^2-r_1^2)d\theta=EGJF$, which is equivalent to the sectorial strip $EGHI$ of the given figure; the second is the sum of such strips for the whole area. The final result is the limit of the sum of such figures as PQ when both $d\theta$ and $dr \doteq 0$.

Observe that the element $r\,d\theta\,dr=PR\,.\,PS$ as in Fig. 106.

Ex. 1. The area (Fig. 108) between the circles $r=2a\cos\theta$, $r=2b\cos\theta$, $(b>a)$, is

$$2\int_0^{\frac{\pi}{2}}\int_{2a\cos\theta}^{2b\cos\theta} r\,d\theta\,dr=\pi(b^2-a^2)$$

2. Find the area (Fig. 109) bounded by the curves $\theta=r^3+r$, $\theta=r^3-r$, $r=1$.

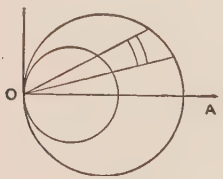

Fig. 108

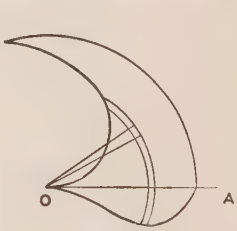

Fig. 109.

[Integrate first with regard to θ.] *Ans.* $\frac{2}{3}$.

3. The moment of inertia of the circle $r=2a\cos\theta$ with reference to a normal axis through the polar origin

$$=2\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^3 d\theta\,dr=\tfrac{3}{2}\pi a^4.$$

4. Find the moment of inertia of the lemniscate $r^2=a^2\cos 2\theta$ (Fig. 27) with reference to a normal axis through the origin.

Ans. $\frac{1}{8}\pi a^4$.

5. The *potential* of a particle of mass m at distance R being m/R, show that the potential of a lamina (thickness t, density μ) at a point B on the perpendicular to the lamina through the polar origin O ($OB=c$) is

$$\iint \frac{\mu t r \, d\theta \, dr}{\sqrt{c^2+r^2}}.$$

(1) Find the potential at B if the lamina is the circle $r=a$.

Ans. $2\pi\mu t(\sqrt{c^2+a^2}-c)$.

(2) Find the potential of a circular lamina at a point in the circumference. *Ans.* $4a\mu t$.

6. If a curve revolve about the initial line show that the volume $=2\pi\iint r^2 \sin\theta \, d\theta \, dr$.

Find this volume for a complete revolution of the cardioid $r=a\cos^2\frac{1}{2}\theta$, Fig. 89. *Ans.* $\frac{1}{3}\pi a^3$.

7. Find the moment of inertia of an anchor-ring (see Ex. 14, p. 180) about its axis.

Take as origin the centre of the circle (radius a) to be revolved, and the perpendicular (length b) on the axis of revolution as initial line. The m.i $=2\int_0^{\pi}\int_0^{a} \mu \cdot 2\pi(b-r\cos\theta) \cdot r \, d\theta \, dr \cdot (b-r\cos\theta)^2$

Ans. $m(\frac{3}{4}a^2+b^2)$.

179. Volume of a solid. Rectangular coördinates. Let P be the point (x,y,z) and let PR, PS, PT be dx, dy, dz, infinitesimal increments of x, y, and z. The parallelepiped $PQ=dx\,dy\,dz$ is taken as the element of volume, the limit of the sum of such elements being obtained by successive integration. Thus to find the volume of the solid, Fig. 110, bounded by the coördinate planes and a surface whose equation is given:

$$(1)\quad \left(\int_0^{IU} dz\right) dx\,dy = IU \cdot dx\,dy,$$

$$(2)\quad \left(\int_0^{DG} IU \cdot dy\right) dx = VDG \cdot dx,$$

$$(3)\quad \int_0^{OA} VDG \cdot dx = COBA,$$

the required volume.

The first result is equivalent to the column standing on $IK(=dx\,dy)$, the second to the slice between VDG and WEF and therefore of thickness dx, the third is the sum of such slices.

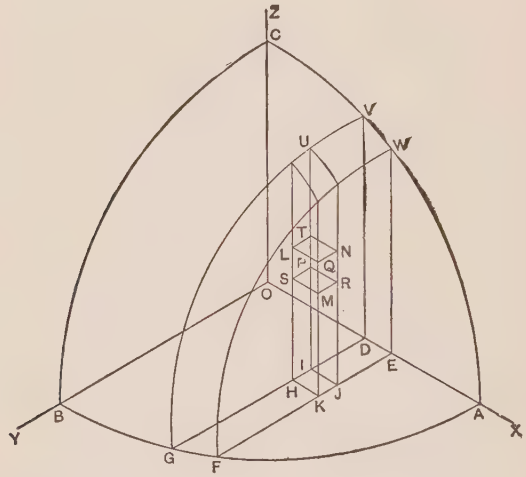

Fig 110.

The whole operation is indicated by the "triple integral"

$$\int_0^a\int_0^{y_1}\int_0^{z_1} dx\,dy\,dz,$$

where z_1 is the z of the given surface, y_1 is the y of the curve AGB in which the surface is cut by the plane $z=0$, and $a=OA$.

Ex. 1. To find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

$\frac{1}{8}V=\int_0^a\int_0^{y_1}\int_0^{z_1} dx\,dy\,dz$, where z_1 is the z of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and y_1 is the y of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, i.e

$$y_1=\frac{b}{a}(a^2-x^2)^{\frac{1}{2}} \quad \text{and} \quad z_1=\frac{c}{b}(y_1^2-y^2)^{\frac{1}{2}}.$$

Since $$\int_0^{z_1} dz = z$$

$$\therefore \tfrac{1}{8}V = \int_0^a \int_0^{y_1} z_1 \, dx \, dy = \int_0^a \int_0^{y_1} \left[\frac{c}{b}(y_1^2 - y^2)^{\frac{1}{2}} dy \right] dx$$

$$= \int_0^a \frac{\pi c}{4b} y_1^2 dx = \int_0^a \frac{\pi bc}{4a^2}(a^2 - x^2) dx = \frac{\pi abc}{6}.$$

$$\therefore V = \tfrac{4}{3}\pi abc.$$

$z_1 \, dx \, dy$ is equivalent to a column, $\frac{\pi c}{4b} y_1^2 \, dx$ to a slice, and $\frac{\pi abc}{6}$ is the sum of the slices for an octant.

2. Find $\iiint \mu(y^2 + z^2) dx \, dy \, dz$ for an ellipsoid, i.e., the moment of inertia with reference to the x-axis.

$$8\int_0^a \int_0^{y_1} \int_0^{z_1} \mu(y^2 + z^2) dx \, dy \, dz = 8\mu \int_0^a \int_0^{y_1} (y^2 z_1 + \tfrac{1}{3} z_1^3) dx \, dy$$

$$= \frac{\pi \mu c}{2b^3}(b^2 + c^2) \int_0^a y_1^4 \, dx = \tfrac{4}{15}\pi abc\mu(b^2 + c^2) = m\frac{b^2 + c^2}{5}.$$

3. Find the volume of the hyperbolic paraboloid $az = x^2 - y^2$ in the first octant, x varying from 0 to h.

$$\int_0^h \int_0^x \int_0^{\frac{x^2 - y^2}{a}} dx \, dy \, dz = \frac{1}{6}\frac{h^4}{a}.$$

4. Find the volume of $x^2 - y^2 = x^2 z^2$ in the first octant, x varying from 0 to h. *Ans.* $\frac{1}{8}\pi h^2$

5. In Ex. 3 find the moment of inertia with reference to the z-axis. *Ans.* $m\frac{4}{5}h^2$

6. A rectangular parallelepiped which has its base in the xy-plane and its sides parallel to the other coördinate planes, intersects the hyperbolic paraboloid $az = xy$. Show that the enclosed volume = base × mean length of the vertical edges.

7. Find $\iiint xy \, dx \, dy \, dz$ and $\iiint xyz \, dx \, dy \, dz$ for the octant of an ellipsoid. *Ans.* (1) $\frac{a^2b^2c}{1 \cdot 3 \cdot 5}$, (2) $\frac{a^2b^2c^2}{2 \cdot 4 \cdot 6}$.

8. Find the volume enclosed in the first octant by the coördinate planes and the surface $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1$.

Ans. $\frac{abc}{90}$.

180. Volumes. Polar coördinates. Let O be the origin, OA the initial line, $BAOC$ the initial plane. A plane revolving through the angle ϕ about OA from the initial plane contains a point P, $OP=r$ making an angle θ with OA. The polar coördinates of P are ϕ, θ, r; r and θ are the ordinary polar coördinates of plane geometry, and (§ 178) $PM = r\,d\theta\,dr + I_1$, where I_1 is a higher infinitesimal. The increment $d\phi$ brings P to T and PM to TQ, $PT = PA\,d\phi = r\sin\theta\,d\phi$. Hence the element of volume PQ

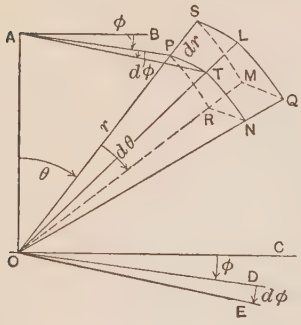

Fig. 111.

$$= (r\,d\theta\,dr + I_1)(r\sin\theta\,d\phi + I_2) = r^2\sin\theta\,d\phi\,d\theta\,dr + I_3.$$

$$\therefore\ V = \iiint r^2\sin\theta\,d\phi\,d\theta\,dr.$$

(The element is, as in Fig. 110, $PR\,.\,PS\,.\,PT$.)

Ex. 1. Volume of a sphere. (1) The origin being the centre,

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2\sin\theta\,d\phi\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^{\pi} \tfrac{1}{3}a^3\sin\theta\,d\phi\,d\theta = \int_0^{2\pi} \tfrac{2}{3}a^3\,d\phi = \tfrac{4}{3}\pi a^3.$$

The first integral $\frac{1}{3}a^3\sin\theta\,d\phi\,d\theta$ = a pyramid with vertex at the centre and base on the surface of the sphere, the second $\frac{2}{3}a^3 d\phi$ = a wedge of which the angle is $d\phi$ and the edge a diameter of the sphere, the third $\frac{4}{3}\pi a^3$ = the sum of the wedges.

(2) If the origin is on the surface, and the initial line a diameter,

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2\sin\theta\,d\phi\,d\theta\,dr.$$

(3) If the origin is a point on the surface, the initial line a tangent, and the initial plane passes through the centre,

$$V=4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{2a\sin\theta\cos\phi} r^2\sin\theta\,d\phi\,d\theta\,dr.$$

2. The vertex of a cone of vertical angle 2α is on the surface of a sphere of radius a, and its axis passes through the centre of the sphere. Find the common volume. *Ans.* $\frac{4}{3}\pi a^3(1-\cos^4\alpha)$.

3. The moment of inertia of a sphere about a diameter

$$=2\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a}\mu r^4\sin^3\theta\,d\phi\,d\theta\,dr=m\,\tfrac{2}{5}a^2,$$

and about a tangent line

$$=4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{2a\sin\theta\cos\phi}\mu r^4\sin^3\theta\,d\phi\,d\theta\,dr=m\,\tfrac{7}{5}a^2.$$

4. The potential of a solid sphere (density μ) at a point on the surface $=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta}\mu r\sin\theta\,d\phi\,d\theta\,dr=\frac{4}{3}\mu\pi a^2=\frac{m}{a}$.

5. To find the potential V of a spherical shell of infinitesimal thickness (radius r, thickness t, density μ) at any point A.

Take O, the centre of the sphere, as origin, OA as initial line, and let $OA=c$. Then

$$V=\int_0^{2\pi}\int_0^{\pi}\frac{\mu r^2\sin\theta\,d\phi\,d\theta\,.\,t}{\sqrt{c^2+r^2-2cr\cos\theta}},\text{ where } r \text{ is constant.}$$

$$\therefore\ V=\frac{2\pi\mu rt}{c}[(c+r)-(c-r)],\quad c>r,$$

and $$=\frac{2\pi\mu rt}{c}[(r+c)-(r-c)],\quad c<r.$$

$$\therefore\ V=\frac{4\pi\mu r^2t}{c}=\frac{m}{c}\text{ for } A \text{ outside the sphere,}$$

and $$=4\pi\mu rt=\frac{m}{r}\text{ for } A \text{ inside the sphere.}$$

In the former case the potential is the same as if the whole mass were at the centre of the sphere, in the latter it is independent of c, and therefore the same at all points inside the sphere

6. To find the potential of a homogeneous solid sphere (radius a) at any point A.

Consider the sphere as made up of concentric shells of infinitesimal thickness $t=dr$. Then from Ex. 5,

$$V=\frac{4\pi\mu}{c}\int_0^a r^2\,dr=\frac{4}{3}\frac{\pi\mu a^3}{c}=\frac{m}{c},\quad A \text{ outside},$$

and

$$=\frac{4\pi\mu}{c}\int_0^c r^2\,dr+4\pi\mu\int_c^a r\,dr=2\pi\mu(a^2-\tfrac{1}{3}c^2),\quad A \text{ inside}.$$

181. Volume. Mixed coördinates. A volume $=\int\int z\,dx\,dy$ if the base is in the xy-plane. Instead of $dx\,dy$ for the base of the column of height z we may take the polar element of area $r\,d\phi\,dr$. Then

$$V=\int\int zr\,d\phi\,dr.$$

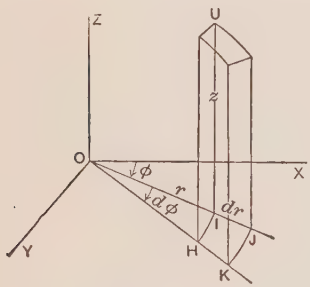

Fig. 112.

Ex 1. For the volume of a sphere by this method,

$$\tfrac{1}{2}V=\int_0^{2\pi}\int_0^a\sqrt{a^2-r^2}\;r\,d\phi\,dr$$

$$=\int_0^{2\pi}\tfrac{1}{3}a^3\,d\phi=\tfrac{2}{3}\pi a^3.$$

The first inetgral $\frac{1}{3}a^3\,d\phi$ is a wedge whose edge is OZ and angle $d\phi$, the second is the sum of such wedges for a hemisphere.

2. The axis of a right circular cylinder of radius b passes through the centre of a sphere of radius $a(>b)$. Find the common volume.

$$2\int_0^{2\pi}\int_0^b\sqrt{a^2-r^2}\;r\,d\phi\,dr=\tfrac{4}{3}\pi\,[a^3-(a^2-b^2)^{\frac{3}{2}}].$$

3. A right circular cylinder of diameter a penetrates a sphere of radius a, the centre of the sphere being on the surface of the cylinder. Find the common volume.

$$4\int_0^{\frac{\pi}{2}}\int_0^{a\cos\phi}\sqrt{a^2-r^2}\,r\,d\phi\,dr=\tfrac{2}{3}(3\pi-4)a^3.$$

4. The axes of two equal right circular cylinders of radius a intersect at right angles. Find the common volume.

$$8\int_0^{\frac{\pi}{2}}\int_0^{a}\sqrt{a^2-r^2\sin^2\phi}\,r\,d\phi\,dr=\tfrac{8}{3}a^3\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos^3\phi}{\sin^2\phi}\right)d\phi$$

$$=\tfrac{8}{3}a^3\left[\tan\frac{\phi}{2}+\sin\phi\right]_0^{\frac{\pi}{2}}=\tfrac{16}{3}a^3.$$

5. The volume between the surface $z=e^{-(x^2+y^2)}$ and its asymptotic plane $z=0$.

This is a surface of revolution about the z-axis

(1) $V=\int_0^{2\pi}\int_0^{\infty}zr\,d\phi\,dr$, and $x^2+y^2=r^2$.

$$\therefore\ V=2\pi\int_0^{\infty}re^{-r^2}dr=2\pi\left[-\tfrac{1}{2}e^{-r^2}\right]_0^{\infty}=\pi.$$

(2) Also $V=4\int_0^{\infty}\int_0^{\infty}z\,dx\,dy=4\int_0^{\infty}\int_0^{\infty}e^{-x^2}\,.\,e^{-y^2}dx\,dy$

$$=4\int_0^{\infty}e^{-y^2}dy\,.\int_0^{\infty}e^{-x^2}dx\,*=4\left(\int_0^{\infty}e^{-x^2}dx\right)^2.$$

From (1) and (2), $\int_0^{\infty}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$. (Cf. § 124.)

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* If the integration limits of each variable are independent of the other we evidently have

$$\int\int f(x)\,F(y)dx\,dy=\int f(x)dx\,.\int F(y)dy.$$

182. Area of any surface. Let the parallelepiped, Fig. 42, which has the base $dx\,dy$ in the xy-plane and its sides parallel to the z-axis, intersect the tangent plane at $P(x, y, z)$ and the surface in sections of area A_t and A_s, respectively, which are assumed to be equivalent infinitesimals. Let α, β, γ be the direction angles of the normal at P. Then

$$dx\,dy = A_t \cos\gamma, \quad \therefore\ A_s = \sec\gamma\,dx\,dy + I,$$

where I is a higher infinitesimal. Hence the surface S

$$= \int \sec\gamma\,dx\,dy = \int \sec\alpha\,dy\,dz = \int \sec\beta\,dz\,dx.$$

$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are proportional (§ 63) to $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ if $u = c$ is the equation of the surface, and hence to 1, $-\frac{\partial z}{\partial x}$, $-\frac{\partial z}{\partial y}$ if the equation is in the form $z = f(x, y)$. Hence

$$\sec\gamma = \frac{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2}}{\frac{\partial u}{\partial z}},$$

or

$$= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$

Ex. 1 A right circular cylinder of diameter a penetrates a sphere of radius a, the centre of the sphere being on the surface of the cylinder. Find the surface of the sphere which is inside the cylinder.

Let the equations be $x^2 + y^2 + z^2 = a^2$ (1), $x^2 + y^2 = ax$ (2).

For (1) $\sec\gamma = \frac{a}{z} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$.

$$\therefore S=4\int_0^a\int_0^{\sqrt{ax-x^2}}\frac{a\,dx\,dy}{\sqrt{a^2-x^2-y^2}}=4a\int_0^a\sin^{-1}\sqrt{\frac{x}{a+x}}dx\ *$$

$$=4a\left[(x+a)\sin^{-1}\sqrt{\frac{x}{a+x}}-\sqrt{ax}\right]_0^a=2(\pi-2)a^2.$$

2. In Ex. 1 find the volume of the cylinder which is inside the sphere.

For (2), $\sec\beta=\frac{\frac{1}{2}a}{y}=\frac{a}{2\sqrt{ax-x^2}}$, and (1) and (2) intersect in $z^2=a^2-ax$.

$$\therefore S=4\int_0^a\int_0^{\sqrt{a^2-ax}}\frac{a\,dx\,dz}{2\sqrt{ax-x^2}}=2a\int_0^a\sqrt{\frac{a}{x}}\,dx=4a^2.$$

3. The axes of two equal right circular cylinders of radius a intersect at right angles. Find the cylindrical surface enclosed. Let the equations be $x^2+z^2=a^2$, $y^2+z^2=a^2$. *Ans.* $16a^2$.

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* Let $x/(a+x)=\sin^2\theta$.

CHAPTER XXXVI.

MEAN VALUES.

183. Let the base $b-a$ of the curve, Fig. 113, be divided into n equal parts, at the extremities of which ordinates $y_1, y_2, \ldots$ are drawn. The limit of

$$\frac{y_1+y_2+\ldots+y_n}{n}$$

for n infinite is called the *mean value* of y for the interval a to b of x. If $dx=$ the length of the equal segments of $b-a$, $n=(b-a)/dx$. Hence the mean value of y is

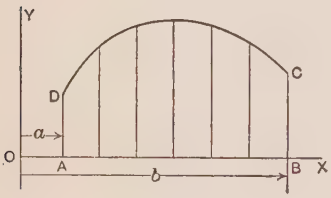

FIG. 113.

$$\pounds\Sigma y\frac{dx}{b-a}=\frac{\int_a^b y\,dx}{b-a}. \qquad (1)$$

Since the numerator = the area $ABCD$, the mean ordinate is equal to the height of a rectangle which has the same base and area as the given figure.

The result (1) may be regarded as the mean value, for the interval a to b of the variable, of any function which is single-valued and continuous for that interval.

Ex. 1 The mean ordinate of a semicircle of radius $a=\frac{\frac{1}{2}\pi a^2}{2a}=\frac{\pi a}{4}$ $=\cdot 7854a$.

2. The mean square of the ordinate of a semicircle

$$= \frac{\int_{-a}^{a} y^2\,dx}{2a} = \frac{\int_{-a}^{a} (a^2 - x^2)dx}{2a} = \tfrac{2}{3}a^2.$$

3. Find the mean ordinate and the mean square of the ordinate of the curve $y = a \sin nx$ from $x = 0$ to $x = \pi$.

Ans. (1) $2a/n\pi$, (2) $a^2/2n$.

4. The arc of a semicircle is divided into equal parts, from the extremities of which perpendiculars are drawn to the diameter. What is the mean value of their length?

Radii through the points of section divide the angle at the centre into equal parts. Hence

$$\pounds \Sigma a \sin\theta \div \frac{\pi}{d\theta} = \frac{a}{\pi}\int_0^{\pi} \sin\theta\, d\theta = \frac{2a}{\pi}.$$

5. In Ex. 4 find the mean distance of the points in the circumference from one end of the diameter. *Ans.* $4a/\pi$.

6. A straight line AB of length a is divided into equal parts, P being a point of section. Find the mean value of $AP \,.\, PB$ for all positions of P. *Ans.* $\frac{1}{6}a^2$.

7. The northern hemisphere is divided into zones of equal area. Find their mean distance from the pole.

Ans. Arc = radius.

8. A rectangle is divided into rectangles by lines which divide the sides equally. Find the mean square of the distance of the rectangles from one corner of the given rectangle.

The sides at that corner being axes,

$$\pounds \Sigma (x^2 + y^2) \div \frac{ab}{dx\,dy} = \frac{1}{ab}\int_0^a \int_0^b (x^2 + y^2)dx\,dy = \tfrac{1}{3}(a^2 + b^2).$$

184. Let $d\lambda$ be an element of any quantity (length, area, volume, mass, time, etc.), and u a variable which is taken any number m times per unit of λ. Then

$$\frac{\pounds \Sigma\, um\, d\lambda}{\pounds \Sigma\, m\, d\lambda} = \frac{\pounds \Sigma\, u\, d\lambda}{\pounds \Sigma\, d\lambda}$$

expresses the limit of the sum of the u's divided by their number, and is thus the mean value of u for the range involved in the summation. If the elements are unequal the result is still the same as the mean value of u taken once for each of the elements if they were equal, since 1 for each one-mth part of the unit is equivalent to m per unit.

Ex 1. To find the mean distance of points within a circle of radius a from a given point on the circumference.

In this case $d\lambda$ is an element of area, say $r\,d\theta\,dr$ (§ 178), u is r, hence the mean value

$$= \frac{2\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^2 d\theta\, dr}{\pi a^2} = \frac{32}{9\pi} a.$$

2. The plane base of a hemisphere of radius a is horizontal. Find (1) the mean height of points within the hemisphere (the element being one of volume), (2) the mean height of points on the curved surface, (3) the mean depth of points in the base below the curved surface. *Ans.* (1) $\frac{3}{8}a$, (2) $\frac{1}{2}a$, (3) $\frac{2}{3}a$.

In (1) and (2) the mean height is the height of the centre of gravity,* in (3) it is volume/base.

3. Find the mean square of the distance of points within a sphere of radius a from (1) the centre, (2) a point on the surface, (3) a diameter. *Ans.* (1) $\frac{3}{5}a^2$, (2) $\frac{8}{5}a^2$, (3) $\frac{2}{5}a^2$.

---

* The point whose coördinates are the mean values of the rectangular coördinates of points in a body for equal elements of mass is easily seen to be the centre of mass (or of gravity) and therefore the centroid if the body is homogeneous. The centroid is sometimes called the centre of mean position.

CHAPTER XXXVII.

INTRINSIC EQUATION OF A CURVE. THE TRACTRIX THE CATENARY.

Intrinsic Equation of a Curve.

185. Let the tangent or normal of a curve turn through an angle λ while the point of contact moves a distance s along the curve. The equation connecting s and λ is called the intrinsic equation of the curve.

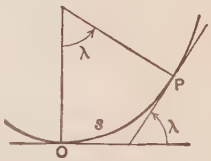

FIG. 114.

Ex. 1. The intrinsic equation of a circle of radius a is obviously $s = a\lambda$.

2. To find the intrinsic equation of the semi-cubical parabola $ay^2 = x^3$ (Fig. 29), the intrinsic origin being the cusp.

The tangent at the origin is the x-axis. Hence

$$\tan\lambda = \frac{dy}{dx} = \frac{3}{2}\left(\frac{x}{a}\right)^{\frac{1}{2}}.$$

$\therefore\ x = \frac{4}{9}a\tan^2\lambda$, whence $y = \frac{8}{27}a\tan^3\lambda$.

Also $$ds = \sqrt{dx^2 + dy^2} = \tfrac{8}{9}a\sec^3\lambda\tan\lambda\, d\lambda.$$

$$\therefore\ s = \tfrac{8}{9}a\int_0^{\lambda}\sec^3\lambda\tan\lambda\, d\lambda, \quad \text{or} \quad s = \tfrac{8}{27}a(\sec^3\lambda - 1),$$

the equation required.

3. Find the intrinsic equation of the common parabola $y^2 = 4ax$, the origin being the vertex

If the tangent make an angle λ with the positive direction of the y-axis, $\tan\lambda = dx/dy$.

Ans. $s = a\sec\lambda\tan\lambda + a\log(\sec\lambda + \tan\lambda)$.

4. Find the intrinsic equation of the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ (Fig. 19), the origin being at a cusp (say, at $x=0$, $y=0$).

[Show that $\lambda(=YSP)=\frac{1}{2}\theta$.] *Ans.* $s=4a(1-\cos\lambda.)$

5. Find the intrinsic equation of the four-cusped hypocycloid $x=a\sin^3\theta$, $y=a\cos^3\theta$ (Fig. 18), the origin being at a cusp (say at $x=0$, $y=a$).

[Show that $\lambda(=OST)=\theta$.] *Ans.* $s=\frac{3}{4}a(1-\cos 2\lambda)$.

6. Show that an equation may be transformed to a new origin at (s', λ') by substituting $s+s'$ for s and $\lambda+\lambda'$ for λ in the given equation.

7. Find the equations of the curves of Exs. 4 and 5 when the origin is a vertex (the middle point of the arc between successive cusps). *Ans.* (1) $s=4a\sin\lambda$, (2) $s=\frac{3}{4}a\sin 2\lambda$.

186. Instead of expressing x and y in terms of λ, we may be able to express s and λ in terms of x or some other variable, and eliminate that variable.

Ex. 1. The catenary $y=a\cosh(x/a)$ (Fig. 117), the origin being the lowest point, and hence the initial tangent being parallel to the x-axis.

$\tan\lambda=dy/dx=\sinh(x/a)$, and $ds=\cosh(x/a)\,dx$, whence $s=a\sinh(x/a)$. $\therefore\ s=a\tan\lambda$.

2. The cardioid $r=a(1-\cos\theta)$, the cusp being the polar and the intrinsic origin.

$$ds=\sqrt{r^2d\theta^2+dr^2}=2a\sin\tfrac{1}{2}\theta\,d\theta,\quad \therefore\ s=4a(1-\cos\tfrac{1}{2}\theta).$$

Also $\lambda=\theta+\phi$, where (§ 136) $\tan\phi=r\,d\theta/dr=\tan\frac{1}{2}\theta$.

$\therefore\ \lambda=\frac{3}{2}\theta$. Hence $s=4a(1-\cos\frac{1}{3}\lambda)$.

Show that the equation is $s=4a\sin\frac{1}{3}\lambda$ if the origin is the point most remote from the cusp.

3. Show that the intrinsic equation of an epicycloid is

$$s=\frac{4b(a+b)}{a}\left(1-\cos\frac{a\lambda}{a+2b}\right),$$

when the origin is a cusp, and

$$s=\frac{4b(a+b)}{a}\sin\frac{a\lambda}{a+2b},$$

when the origin is a vertex.

187. The radius of curvature. The curvature is the s-rate of λ (§ 86) and hence $R=ds/d\lambda=f'(\lambda)$ if $s=f(\lambda)$ is the equation of the curve.

188. The evolute. Let O be the origin of the given curve $s=f(\lambda)$, P any other point on the curve, C, Q the centres of curvature of O and P, and let C be taken as the origin of the evolute. Then $PQ=f'(\lambda)$ and $OC=f'(0)$, also $CQ=PQ-OC$ (§ 90 (B)). Hence the equation of the evolute is

$$s=f'(\lambda)-f'(0).$$

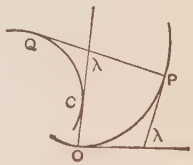

FIG. 115.

Ex. 1. Show that the evolute (1) of a parabola is a semi-cubical parabola, (2) of a cycloid is an equal cycloid, (3) of a four-cusped hypocycloid is a similar curve of twice the size of the given curve, (4) of a cardioid is a cardioid of one third the size of the given one.

2. What is the intrinsic equation of the involute of (1) a circle, (2) a catenary, the involute beginning at a point on the given curve? *Ans.* (1) $s=\frac{1}{2}a\lambda^2$, (2) $s=a\log\sec\lambda$.

The Tractrix.

189. This is the curve in which the tangent is of constant length.

Let (x, y) be the coördinates of a point P on the curve (Fig. 116), and let the tangent $PT=a$.*

From the figure $dy/dx=-y/\sqrt{a^2-y^2}$, from which the equation may be found by integrating, the result being

$$x=a\,\mathrm{sech}^{-1}(y/a)-\sqrt{a^2-y^2}.$$

* The curve is the path of a body which is drawn along on a rough horizontal plane by a string of length a, the other end of which is moved along a straight line OX; whence the name of the curve.

Since $ydx = -dy\sqrt{a^2-y^2}$, the element of the area of the curve = the element of the area of the circle of radius a, $\therefore$ the whole area between the curve and its asymptote (the x-axis) is the same as that of the circle, viz., πa^2.

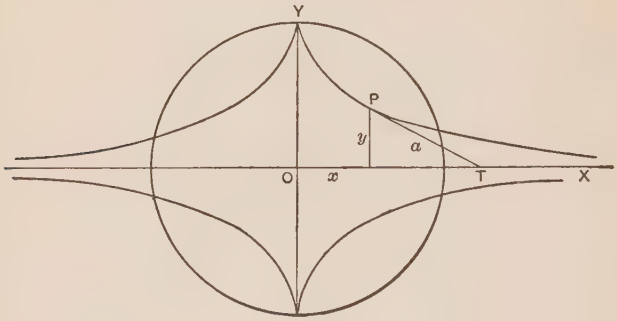

Fig. 116

The length of the curve from Y to any point whose ordinate is b may be found as follows:

$$\frac{ds}{dy} = -\frac{a}{y}, \quad \therefore \; s = -a\int_a^b \frac{dy}{y} = a \log\left(\frac{a}{b}\right).$$

Let the tangent make an angle λ with YO. Then $s = a \log \sec \lambda$, which is the intrinsic equation (origin Y) of the curve. Hence the radius of curvature $= ds/d\lambda = a \tan \lambda$, and the evolute is (§ 188) $s = a \tan \lambda$, a catenary (§ 192).

The area of the surface of revolution * of the curve about the x-axis $= 4\pi a^2$, and the volume $= \frac{2}{3}\pi a^3$.

* This surface is known as the *pseudo-sphere*.

The Catenary.

190. This is the curve formed by a uniform chain hanging vertically.

Let A be the lowest point, P any other point. From P draw PB vertically and equal to the length AP or s of the chain, and from B draw a horizontal line to meet the tangent at P in C, and let $BC=a$.

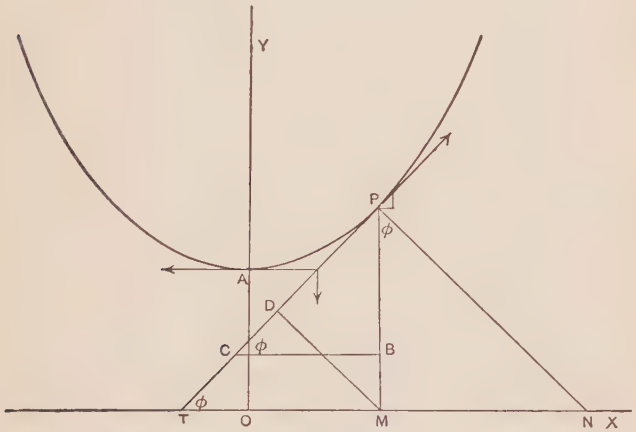

Fig 117.

191. Mechanics of the figure. The portion AP of the chain is in equilibrium under the action of three forces, viz., the horizontal tension at A, the tension at P in the direction of the tangent, and the weight, which is vertical. Hence PBC is a triangle of forces for AP, and since the vertical force on AP is the weight of a length PB of the chain, it follows that the tension at P is equal to the weight of a length CP of the chain, and the tension at A to the weight of a length a of the chain, and therefore a is constant.

192. Geometry of the figure. Draw from A a vertical line and take $OA=a$; take O as the origin, OA as y-axis and a

horizontal line OX as x-axis. Then $OM=x$, $MP=y$. (By this choice of axes the constants of integration will $=0$.)

Since CB, BP, and CP are a, s, and $\sqrt{a^2+s^2}$, respectively, we have

$$\frac{dy}{dx}=\frac{s}{a}\ (1),\quad \frac{dx}{ds}=\frac{a}{\sqrt{a^2+s^2}}\ (2),\quad \frac{dy}{ds}=\frac{s}{\sqrt{a^2+s^2}}\ (3).$$

From (3) $dy=s\,ds/\sqrt{a^2+s^2}$, $\therefore\ y=\sqrt{a^2+s^2}=CP$, (4)
$\therefore$ the tension at any point P is equal to the weight of a length y of the chain.

From (2), $dx=a\,ds/\sqrt{a^2+s^2}$, $\therefore\ x=a\,\sinh^{-1}(s/a)$,

$$\therefore\ s=a\sinh\frac{x}{a},\quad \text{or}\quad =\frac{a}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right), \tag{5}$$

which gives the length from the lowest point to the point whose abscissa is x.

From (1) and (5), $dy=\sinh\frac{x}{a}dx$.

$$\therefore\ y=a\cosh\frac{x}{a},\quad \text{or}\quad =\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right), \tag{6}$$

the equation of the curve.

From (1), $s=a\tan\phi$, the intrinsic equation (origin A) of the curve.

The normal. $NP:MP::CP:CB$, $\therefore\ NP/y=y/a$, or $NP=y^2/a$.

The radius of Curvature. $R=ds/d\phi=a\sec^2\phi=ay^2/a^2=y^2/a$, or the radius of curvature is equal to the normal.

Let D be the foot of the perpendicular from M on PT. Since $MP=CP$, $\therefore\ MD=CB=a$ and $DP=BP=s$.

Hence the locus of D is the involute of the catenary.

Also MD, the tangent at D to the involute, is of constant length, $\therefore$ the involute is a tractrix.

The intrinsic equation of the evolute is $s=a(\sec^2\phi-1)$.

CHAPTER XXXVIII.

INFINITE SERIES.*

193. A series is a succession of terms which follow one another according to some law. The series is said to be infinite when it does not terminate. If we add the terms of the infinite series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ it is seen that the sum approaches a limit, viz., 2; for this reason the series is said to be *convergent*. That the limit is 2 is also seen by taking the sum of the first n terms, which is $2-\frac{1}{2^{n-1}}$, and finding the limit of this when $n=\infty$.

This series is a particular case of the infinite geometrical progression

$$1+x+x^2+x^3+\ldots \qquad (1)$$

From elementary algebra the limit of the sum is $1/(1-x)$ if $|x|<1$. If $|x|\geqq 1$ the series has no limit or is non-convergent.

194. Let $u_0+u_1+u_2+u_3+\ldots+u_n+u_{n+1}+\ldots$ (2) be an infinite series, and let s_n be the sum of the first n terms. The series is then convergent if s_n has a limit when $n=\infty$; let the limit, if it exists, be s. The limit must as in all other cases be a definite finite quantity (§ 2) such that $s-s_n$ has the limit 0. Hence a series cannot be convergent unless $u_n \doteq 0$ when $n=\infty$, i.e. unless the terms tend to a limit 0. This is a necessary but not a sufficient condition for convergence.

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* On the subject of Infinite Series the student may consult Osgood's *Introduction to Infinite Series* (Harvard University), also Gibson's *Calculus* (Macmillan).

If a series is non-convergent it is either *divergent* or *oscillatory*, divergent if s_n becomes infinite with n, oscillatory if s_n remains finite but does not approach a limit. Series (1) is divergent if $|x|>1$ or $x=1$, and oscillatory if $x=-1$, s_n in the latter case being 1, 0, 1, 0, etc., as n increases, but never approaching a limit.

In general, a series is of no practical value unless it is convergent.

195. Without expressing s_n in terms of n it may be possible to test a given series for convergence. For this purpose various methods are given in works on algebra; we recall a few results which are of importance in our work.

(1) A series is convergent if (*a*) the terms are alternately $+$ and $-$, (*b*) the absolute value of the terms constantly diminishes, and (*c*) the limit of that value is 0 when $n=\infty$.

Ex. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ is convergent (see § 2, Ex. 3). The limit of the sum lies between ·69314 and ·69315. It $=\log_e 2$ (§ 197 Ex. 1). The series $2-\frac{2}{2}+\frac{4}{3}-\ldots$ satisfies conditions (*a*) and (*b*) but not (*c*). It is oscillatory (see § 2, Ex. 4).

(2) The series $1+\frac{1}{2^c}+\frac{1}{3^c}+\ldots$ is convergent when $c>1$, divergent when $c\leqq 1$. Thus the "harmonic series" $1+\frac{1}{2}+\frac{1}{3}+\ldots$ is divergent, but $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ is convergent.

Ex. 1. $\frac{1}{a^2+1^2}+\frac{1}{a^2+2^2}+\ldots$ is convergent, since each term is less than the corresponding term of $1+\frac{1}{2^2}+\ldots$

2. $\frac{1}{a+1}+\frac{1}{a+2^2}+\ldots$, $a>0$, is divergent. For it $>\frac{1}{b+1}+\frac{1}{b+2}+\ldots$, where b is an integer larger than a, and this is a part of the divergent series $1+\frac{1}{2}+\frac{1}{3}+\ldots$

(3) If a series is convergent when all the terms are positive, it will be convergent if any of the terms are negative.

Ex. $\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\ldots$ is convergent for all values of x, for if all the terms were positive they would not be greater than the corresponding terms of $\frac{1}{1^2}+\frac{1}{2^2}+\ldots$

The contrary is not necessarily true, that is, a series may be convergent when some of the terms are negative, but not when those terms are made positive. Thus $1-\frac{1}{2}+\frac{1}{3}-\ldots$ is convergent, but $1+\frac{1}{2}+\frac{1}{3}+\ldots$ is divergent.

A series is said to be *absolutely* convergent when the absolute values of the terms form a convergent series, i.e., when the series remains convergent if all the terms are made positive. If not absolutely convergent it is said to be *conditionally* convergent. Thus $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$ is absolutely convergent, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ is conditionally convergent. It is known that the convergence and limit of an absolutely convergent series are independent of the order in which the terms are taken, whereas the terms of a conditionally convergent series may be grouped so as to converge to a different limit (in fact to an assigned limit) or to diverge. For example, arrange the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ as follows:

$$(1-\tfrac{1}{2})-\tfrac{1}{4}+(\tfrac{1}{3}-\tfrac{1}{6})-\tfrac{1}{8}+(\tfrac{1}{5}-\tfrac{1}{10})-\ldots$$

This $=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\ldots=\frac{1}{2}(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots)$, one-half of the original series.

(4) A series $u_0+u_1+\ldots+u_n+u_{n+1}+\ldots$ is absolutely convergent if R, the absolute value of the limit of u_{n+1}/u_n when $n=\infty$, is <1, divergent if $R>1$. It may or may not be convergent if $R=1$.

196. Power series. The most important infinite series are those of the form

$$a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n+a_{n+1}x^{n+1}+\ldots, \qquad (1)$$

which is called a power series in x. The indices are positive ascending integers, and $a_0, a_1, \ldots$ are independent of x.

By assigning values to x any number of series may be formed from a given power series.

Let the absolute value of the limit of a_n/a_{n+1} when $n=\infty$ be r. It follows from § 195 (4) that the power series is absolutely convergent if $|x|<r$ (i.e. if $x>-r$ and $<r$); divergent if $|x|>r$. If $|x|=r$ the series may be convergent or non-convergent. If $a_n/a_{n+1}=\infty$ when $n=\infty$, the series is convergent for all values of x.

Since the limit for $n+1=\infty$ is the same as for $n=\infty$, the value of r may be found equally well from a_{n+1}/a_{n+2} or from a_{n-1}/a_n, or any two successive coefficients.

Ex. 1. $1+x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}+\ldots$

$$a_n=\frac{1}{n},\quad \therefore\ \frac{a_n}{a_{n+1}}=\frac{n+1}{n}=1+\frac{1}{n}\doteq 1 \quad \text{when} \quad n=\infty,\quad \therefore r=1.$$

Hence the series is absolutely convergent if $|x|<1$. It is conditionally convergent if $x=-1$, divergent for all other values of x.

2. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$

$$a_n/a_{n+1}=(n+1)!/n!=n+1=\infty \text{ when } n=\infty.$$

Hence the series is absolutely convergent for all values of x.

3. $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$ and $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$ are parts of the series of Ex. 2, and are therefore absolutely convergent for all values of x.

4. The Binomial Series $1+mx+\frac{m(m-1)}{2!}x^2+\ldots$

$$a_n/a_{n+1}=(n+1)/(m-n)\doteq -1 \text{ when } n=\infty,\ \therefore\ r=1.$$

Hence the series is absolutely convergent if $|x|<1$ whatever the value of m. It may be proved to be absolutely convergent if $|x|=1$ and m is positive, and conditionally convergent if $x=1$ and $-1<m<0$.

5. If the power series $a_0+a_1x+a_2x^2+\ldots$ is convergent when $|x|<r$, the series $a_1+2a_2x+\ldots$ formed from the derivatives of the terms is also convergent when $|x|<r$.

For $$\pounds_{n=\infty}\frac{n}{n+1}\cdot\frac{a_n}{a_{n+1}}=\pounds_{n=\infty}\frac{a_n}{a_{n+1}}|=|r.$$

6. Show also that the series $c+a_0x+\frac{1}{2}a_1x^2+\frac{1}{3}a_2x^3+\ldots$ formed by integrating (c being a constant) is convergent if $|x|<r$.

197. A power series $a_0+a_1x+a_2x^2+\ldots$ may be such that the limit of the sum for every value of x within the interval of convergence $(-r<x<r)$ is equal to the value of some function $f(x)$ for that value of x. On this understanding we may write

$$f(x)=a_0+a_1x+a_2x^2+\ldots \tag{1}$$

Thus if $|x|<1$, $(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\ldots$, and as a particular case $1/(1+x)=1-x+x^2-\ldots$. If $|x|>1$ there is no connection between the value of the function and the sum of the terms of the series.

If we differentiate both sides of (1) as if the series were finite we have

$$f'(x)=a_1+2a_2x+3a_3x^2+\ldots \tag{2}$$

Similarly, multiplying (1) by dx and integrating,

$$F(x)=c+a_0x+\tfrac{1}{2}a_1x^2+\tfrac{1}{3}a_2x^3+\ldots, \tag{3}$$

c being the integration constant, viz., the value of $F(x)$ when $x=0$. It may be proved that these results hold true (i.e., the new functions are equal to the limits of the sum of the terms of the new series) for all values of x for which the new series are convergent. These values are $-r<x<r$ (§ 196, Exs. 5, 6), the same as those of the original series. If, however, any one of the series is also convergent when $x=r$ or $-r$, it must not be assumed that the others are also convergent for those values. Integration in general increases the rapidity of the convergence of a series, and it may change a series which is divergent when $x=r$ into one which is convergent.

Thus $1+\frac{1}{2}x+\frac{1}{3}x^2+\ldots$ is divergent when $x=1$, but $x+\frac{1}{2^2}x^2+\frac{1}{3^2}x^3+\ldots$ is convergent when $x=1$.

Ex. 1. *Logarithmic series.* If $|x|<1$,

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots$$

Multiply by dx and integrate. Then if $|x|<1$,

$$\log(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\ldots \qquad (1)$$

(No constant of integration, since both sides vanish with x.) (1) is also convergent if $x=1$, § 195 (1), $\therefore \log 2=1-\frac{1}{2}+\frac{1}{3}-\ldots$

(1) may be used for the calculation of the Napierian logarithm of any number >-1 and $\leqq 1$, but it converges too slowly to be of much value for such calculation, unless $|x|$ is very small. Change x in (1) into $-x$ and subtract from (1). Then

$$\log\left(\frac{1+x}{1-x}\right)=2(x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\ldots). \qquad (2)$$

Let $y=(1+x)/(1-x)$, then $x=(y-1)/(y+1)$ Substituting in (2),

$$\log y=2\left[\left(\frac{y-1}{y+1}\right)+\tfrac{1}{3}\left(\frac{y-1}{y+1}\right)^3+\tfrac{1}{5}\left(\frac{y-1}{y+1}\right)^5+\ldots\right]. \qquad (3)$$

This series may be used for the calculation of any Napierian logarithm, since $(y-1)/(y+1)$ is necessarily a proper fraction when y is any positive quantity.

Thus if $y=2$, $(y-1)/(y+1)=\frac{1}{3}$,

$$\therefore \log_e 2=2\left[\tfrac{1}{3}+\tfrac{1}{3}(\tfrac{1}{3})^3+\tfrac{1}{5}(\tfrac{1}{3})^5+\ldots\right]=\cdot 693147.$$

Similarly $\log_e 3=2\left[\frac{1}{2}+\frac{1}{3}(\frac{1}{2})^3+\frac{1}{5}(\frac{1}{2})^5+\ldots\right]=1\cdot 098612.$

Also, $\log_e 4=2\log_e 2=1\cdot 386294.$

Another series may be derived from (2) thus: put $(1+x)/(1-x)=(1+y)/y$, then $x=1/(1+2y)$; hence, remembering that $\log[(1+y)/y]=\log(1+y)-\log y$,

$$\log(1+y)=\log y+2\left[\frac{1}{1+2y}+\tfrac{1}{3}\left(\frac{1}{1+2y}\right)^3+\tfrac{1}{5}\left(\frac{1}{1+2y}\right)^5+\ldots\right].$$

Thus if $y=4$, $\log_e 5=\log_e 4+2\left[\frac{1}{9}+\frac{1}{3}\left(\frac{1}{9}\right)^3+\frac{1}{5}\left(\frac{1}{9}\right)^5+\ \ldots\right]=1{\cdot}609438$. Hence $\log_e 10=\log_e 5+\log_e 2=2{\cdot}302585$, and hence the modulus * of the common logarithms (which $=1/\log_e 10$) is $\cdot 4342945$. The common logarithms may therefore be found from the Napierian logarithms by multiplying by $\cdot 4342945$, and, conversely, the Napierian from the common logarithms by multiplying by $2{\cdot}302585$.

2. *Gregory's series.* Prove that for $-1<x\leq 1$,

$$\tan^{-1}x=x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\ldots \qquad (1)$$

This series gives the radian measure of an angle in terms of its tangent. Show that

$$\frac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots=2\left(\frac{1}{1.3}+\frac{1}{5.7}+\ldots\right).$$

This converges slowly, but by applying (1) to the relation $\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$ a rapidly converging series for the calculation of π is obtained.

Since $\tan^{-1}x=\frac{\pi}{2}-\tan^{-1}\frac{1}{x}$, if $|x|>1$ we have

$$\tan^{-1}x=\frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\ldots$$

3. Prove that

$$\sin^{-1}x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\ldots,\quad |x|<1,$$

and hence, making $x=\frac{1}{2}$, that

$$\pi=3\left(1+\frac{1}{24}+\frac{3}{640}+\frac{5}{7168}+\ldots\right)=3{\cdot}14159\ldots$$

---

* If $x=\log_a y$, then by definition of a logarithm, $a^x=y$. Taking logarithms of this, the base being supposed e, we have $x\log_e a=\log_e y$, $\therefore$ x or $\log_a y=\frac{\log_e y}{\log_e a}$, or the logarithm of y is changed from base e to base a by multiplying by $\frac{1}{\log_e a}$, which is called the modulus of the system of base a.

If $y=e$, $\log_a e=\frac{1}{\log_e a}$, hence the modulus of the common system is also equal to $\log_{10} e$.

Show also that $\sec^{-1}x=\frac{\pi}{2}-\frac{1}{x}-\frac{1}{6x^3}-\frac{1\cdot 3}{2\cdot 4\cdot 5x^5}-\ldots,\quad |x|>1.$

198. Maclaurin's Series. If there is a power series * which $=f(x)$, that series is $f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots$

For, suppose it to be $a_0+a_1x+a_2x^2+\ldots$ Then

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots \tag{1}$$

Differentiating successively,

$$f'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots \tag{2}$$

$$f''(x)=2a_2+2\cdot 3a_3x+3\cdot 4a_4x^2+\ldots \tag{3}$$

$$f'''(x)=2\cdot 3a_3+2\cdot 3\cdot 4a_4x+\ldots \tag{4}$$

etc. If (1) is convergent for $|x|<r$, (2), (3), ... are convergent for the same values of x, and this interval includes $x=0$. Making $x=0$ in (1), (2), (3), ... we have

$$f(0)=a_0,\ f'(0)=a_1,\ f''(0)=2a_2,\ f'''(0)=2\cdot 3a_3,\ldots$$

Substituting in (1),

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots \tag{5}$$

This is Maclaurin's Series. In substituting in (5) we are said to expand or develop $f(x)$ into a power series.

Ex. 1. To expand $\sin x$.

$f(x)=\sin x$, $\quad\therefore\ f(0)=\sin 0=0$;
$f'(x)=\cos x$, $\quad\therefore\ f'(0)=\cos 0=1$;
$f''(x)=-\sin x$, $\quad\therefore\ f''(0)=-\sin 0=0$;
$f'''(x)=-\cos x$, $\quad\therefore\ f'''(0)=-\cos 0=-1$, etc.

Substituting in (5) we have

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots, \tag{6}$$

which gives the sine of an angle in terms of its measure in radians

* There is such a series in most cases if $f(x)$ and all its derivatives are real and finite when $x=0$. See § 210.

The expansion of $\cos x$ may be found in the same manner, or by differentiating (6); hence

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \tag{7}$$

(6) and (7) are convergent for all values of x (§ 196, Ex. 3).

2. The expansions of e^x and a^x are particular cases of Maclaurin's Series. For

$$f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \text{ etc.};$$
$$\therefore f(0) = e^0 = 1, \quad f'(0) = 1, \quad f''(0) = 1, \text{ etc.}$$
$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{8}$$

Similarly, $$a^x = 1 + Ax + \frac{A^2x^2}{2!} + \frac{A^3x^3}{3!} + \dots \quad (A = \log_e a). \tag{9}$$

These series are also convergent for all values of x (§ 196, Ex. 2).

199. If we attempt to expand $\cot x$ by Maclaurin's Series we meet a difficulty at the outset, viz., $\cot 0 = \infty$. This implies that there is no power series for $\cot x$.

Ex. To expand $x \cot x$.

$$x \cot x = x \frac{\cos x}{\sin x} = \frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots} = \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots}.$$

Assume $x \cot x = 1 + a_2x^2 + a_4x^4 + \dots$ (There cannot be any odd powers, since $x \cot x$ is an even function. See Ex. 12, p. 40.) Then

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right)(1 + a_2x^2 + a_4x^4 + \dots).$$

Multiplying * and equating coefficients,

$$x \cot x = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \dots$$

Hence $$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \dots$$

* The product of two convergent series is obtained by multiplying them term by term as if they were finite series, provided that one of them (at least) is absolutely convergent.

200. Formulæ derived from the exponential series. In § 198, (8), write xi for x, where $i \equiv \sqrt{-1}$ (observe that $i^2 = -1$, $i^3 = -\sqrt{-1}$, $i^4 = 1$, $i^5 = i$, etc.).

$$\therefore\ e^{xi} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right),$$

or
$$e^{xi} = \cos x + i \sin x. \tag{1}$$

Writing $-x$ for x,

$$e^{-xi} = \cos x - i \sin x. \tag{2}$$

Adding and subtracting,

$$\cos x = \tfrac{1}{2}(e^{xi} + e^{-xi}), \tag{3}$$

$$\sin x = \frac{1}{2i}(e^{xi} - e^{-xi}), \tag{4}$$

whence
$$\tan x = \frac{1}{i}\left(\frac{e^{xi} - e^{-xi}}{e^{xi} + e^{-xi}}\right) = \frac{1}{i}\left(\frac{e^{2xi} - 1}{e^{2xi} + 1}\right). \tag{5}$$

These are Euler's Formulæ.

In (1) write nx for x.

$$\therefore\ \cos nx + i \sin nx = e^{nxi} = (e^{xi})^n = (\cos x + i \sin x)^n,$$

$$\therefore\ (\cos x + i \sin x)^n = \cos nx + i \sin nx. \tag{6}$$

This is Demoivre's Theorem.

201. Taylor's Series. Suppose the function of x to be $f(h+x)$. The first x-derivative is

$$f'(h+x)\,d(h+x)/dx = f'(h+x).$$

The second is $f''(h+x)$, etc. The values of the function and its derivatives for $x = 0$ are $f(h)$, $f'(h)$, $f''(h)$, ... Hence Maclaurin's Series takes the form

$$f(h+x) = f(h) + f'(h)x + \frac{f''(h)}{2!}x^2 + \frac{f'''(h)}{3!}x^3 + \ldots$$

This is Taylor's Series. The conditions under which the function is represented by the series will be considered in Ch. XXXIX.

Ex. 1. $(h+x)^m = h^m + mh^{m-1}x + \frac{m(m-1)}{2!}h^{m-2}x^2 + \ldots$

2. $\sin(h+x) = \sin h + \cos h \,.\, x - \frac{\sin h}{2!}x^2 - \frac{\cos h}{3!}x^3 + \ldots$

3. If $f(x) = x^3 - 2x^2 - x + 3$, write down $f(x+h)$.

EXAMPLES.

1. Expand $(1+x)^m$, $\log(1+x)$, $\tan^{-1}x$, by Maclaurin's series.

2. $\tan x = x + \frac{x^3}{1\,.\,3} + \frac{2x^5}{3\,.\,5} + \frac{17x^7}{5\,.\,7\,.\,9} + \ldots$

3. $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots$

4. $\log\sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \ldots$

5. $\cos^3 x = 1 - \frac{3x^2}{2} + \frac{7x^4}{8} - \ldots$

6. $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \ldots$

7. $e^x \sec x = 1 + x + x^2 + \frac{2}{3}x^3 + \ldots$

8. $\operatorname{cosec} x = \frac{1}{x} + \frac{x}{3!} + \frac{7x^3}{3\,.\,5!} + \ldots$

9. Show that

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots, \quad \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots,$$

and hence that $\cosh x = \cos ix$, $\sinh x = -i \sin ix$, where $i \equiv \sqrt{-1}$.

10. Assuming $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$, show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}, \quad \text{and} \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots = \frac{\pi^2}{12}.$$

Also that $\int_0^1 \frac{1}{x}\log(1+x)\,dx = \frac{\pi^2}{12}$.

11. Show that $\int_0^1 \frac{dx}{\sqrt{1+x^4}} = 1 - \frac{1}{2}\cdot\frac{1}{5} + \frac{1\,.\,3}{2\,.\,5}\cdot\frac{1}{9} - \ldots$

12. When $x \doteq 0$ show from the series that

(1) $(\sin x)/x \doteq 1$, (2) $(\tan x)/x \doteq 1$, (3) $(1-\cos x)/x^2 \doteq \frac{1}{2}$,
(4) $(\tan x - \sin x)/x^3 \doteq \frac{1}{2}$, (5) $(e^x - 1)/x \doteq 1$.

13. If a circular arc (radius a) subtends an angle θ at the centre, show that when θ is very small

$$\text{arc} - \text{chord} = \tfrac{1}{24} a\theta^3, \text{ nearly.}$$

14. If θ is a small angle, show that

$$\left.\begin{aligned} \sin\theta &= \theta\sqrt[3]{\cos\theta}, \\ \tan\theta &= \theta\sqrt[3]{\cos^{-2}\theta}, \end{aligned}\right\} \text{ nearly.}$$

From these formulæ are derived the rules given in Mathematical Tables for finding the sines and tangents of small angles.

15. The chord of a circular arc is C, the chord of half the arc is c; show that the length of the arc is

$$2c + \tfrac{1}{3}(2c - C), \text{ very nearly.}$$

This formula (Huyghens's) will give $\frac{1}{4}$ of the circumference of a circle of 100 feet radius with an error of less than $1\frac{1}{2}$ inch; it gives $\frac{1}{8}$ of the circumference of the same circle with an error of less than $\frac{1}{50}$ of an inch.

CHAPTER XXXIX.

TAYLOR'S THEOREM.

202. In Ch. XXXVIII the existence of a power series for $f(x)$ or $f(h+x)$ is assumed. We have now to consider under what circumstances this assumption may be justified.

203. Theorem of Mean Value. Let $f(x)$ be a single-valued function, and suppose $f(x)$ and its first derivative $f'(x)$ to be continuous from $x=a$ to $x=b$. The Theorem of Mean Value asserts that

$$\frac{f(b)-f(a)}{b-a}=f'(x_1), \tag{1}$$

where x_1 is some value of x between a and b.

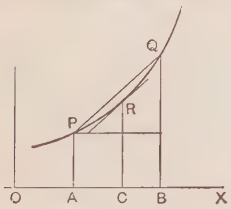


FIG. 118.

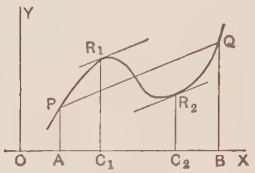


FIG. 119.

Let (Fig. 118) $OA=a$, $OB=b$. If PRQ represents the graph of $f(x)$ from $x=a$ to $x=b$, $AP=f(a)$, $BQ=f(b)$. Hence $\frac{f(b)-f(a)}{b-a}$ is the slope of the straight line PQ. At some point R between P and Q the tangent is parallel to PQ, and the slope of the tangent at R is $f'(x_1)$, where $x_1=OC$. Hence the equality stated in (1). It should be noticed (Fig. 119) that x_1 may have more than one value.

The theorem may not be true if, between $x=a$ and $x=b$, $f(x)$ (Figs. 120, 121) or $f'(x)$ (Figs. 122, 123) has a finite or infinite discontinuity.

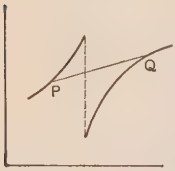


FIG. 120.

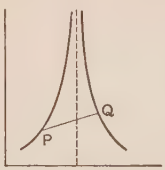


FIG. 121.

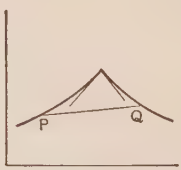


FIG. 122.

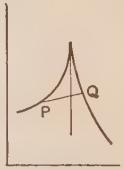


FIG. 123.

204. If, in (1), $f(b)=f(a)$, then $f'(x_1)=0$; i.e., if $f(x)$ and $f'(x)$ are continuous from $x=a$ to $x=b$, and if $f(a)=f(b)$, then $f'(x)=0$ for at least one value between $x=a$ and $x=b$. This is known as Rolle's Theorem.

205. In Fig. 118 let $AB=h$ and $AC=\theta h$, $0<\theta<1$. Then (1) becomes

$$\frac{f(a+h)-f(a)}{h}=f'(a+\theta h),$$

or
$$f(a+h)=f(a)+hf'(a+\theta h).$$

This may be regarded as the beginning of an expansion of $f(a+h)$ in powers of h. We have now to show that the expansion may under certain circumstances be continued to three or more terms.

206. Taylor's Theorem. Let $f(x)$ and its first n derivatives be continuous from $x=a$ to $x=a+h$. Let P be a quantity which is such that

$$f(a+h)-f(a)-f'(a)h-\frac{f''(a)}{2!}h^2-\ \ldots\ -\frac{f^{(n-1)}(a)}{(n-1)!}h^{n-1}=Ph^n. \quad (2)$$

Consider also the following function of x:

$$f(a+h)-f(x)-f'(x)(a+h-x)-\frac{f''(x)}{2!}(a+h-x)^2-\ \ldots$$

$$-\frac{f^{(n-1)}(x)}{(n-1)!}(a+h-x)^{n-1}-P(a+h-x)^n, \quad (3)$$

and let this be called $F(x)$. Differentiating, we obtain

$$F'(x) = -\frac{f^{(n)}(x)}{(n-1)!}(a+h-x)^{n-1} + Pn(a+h-x)^{n-1}. \quad (4)$$

Since $f(x) \ldots f^{(n)}(x)$ are continuous from $x=a$ to $x=a+h$, so also are $F(x)$ and $F'(x)$. Also $F(a)=0$ by (2), and $F(a+h)=0$ by (3); hence (§ 204) $F'(x)=0$ for some value $a+\theta h$ between a and $a+h$. Hence, from (4), $P = \frac{f^{(n)}(a+\theta h)}{n!}$.

Substituting in (2) and transposing,

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{!\,2}h^2 + \ldots + \frac{f^{(n-1)}(a)}{(n-1)!}h^{n-1} + \frac{f^{(n)}(a+\theta h)}{n!}h^n. \quad (5)$$

Hence if $f(x)$ and its first n derivatives are continuous from $x=a$ to $x=a+h$, $f(a+h)$ can be expanded into the finite series (5). This is Taylor's Theorem. It is really a generalization of the Theorem of Mean Value. It should be noted that h is not necessarily positive. The number θ >0 and <1, but its value generally depends upon a, h, and n, as well as the form of the function.

Ex. $\log(a+h) = \log a + \frac{h}{a} - \frac{h^2}{2a^2} + \frac{h^3}{3a^3} - \ldots + \frac{(-1)^{n-1}h^n}{n(a+\theta h)^n}$.

207. If x is written for a, (5) takes the form

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \ldots + \frac{f^{(n)}(x+\theta h)}{n!}h^n.$$

208. The remainder. The last term of (5) is known as Lagrange's form of the remainder (R_n) after n terms. Another form of the remainder R_n (Cauchy's), viz.,

$$\frac{f^{(n)}(a+\theta h)}{(n-1)!}(1-\theta)^{n-1}h^n,$$

may be found by starting with Ph instead of Ph^n in (2).

R_n is the amount of the error when the first n terms of the series are taken as the value of $f(a+h)$. Thus for $\log(a+h)$ (§ 206) the numerical value of the error would lie between $\frac{h^n}{na^n}$ and $\frac{h^n}{n(a+h)^n}$, the greatest and least values of R_n.

The method of making small corrections explained in Ch. XI is equivalent to the use of the first two terms of Taylor's Theorem. In this case the error is therefore $\frac{1}{2}f''(a+\theta h)h^2$, where $0<\theta<1$.

209. Maclaurin's Theorem. Taking $a=0$ in (5) and writing x for h, we have

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+R_n, \qquad (7)$$

where $R_n=\frac{f^{(n)}(\theta x)}{n!}x^n$ or $=\frac{f^{(n)}(\theta x)}{(n-1)!}(1-\theta)^{n-1}x^n$,

according as Lagrange's or Cauchy's form is adopted for the remainder. (7) is the statement of Maclaurin's Theorem. The expansion is therefore possible if $f(x)$ and its first n derivatives are continuous from $x=0$ up to the value of x adopted in the series.

210. Maclaurin's Series. Taylor's Series. If $f(x)$ can be expanded by means of the series (7), and if the values of x are such that $\mathcal{L}_{n=\infty}\ R_n=0$, then

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots,$$

or $f(x)$ is equal to the limit of the sum of the terms of an infinite power series (Maclaurin's Series).

Under similar conditions we obtain from (5)

$$f(a+h)=f(a)+f'(a)h+\frac{f''(a)}{2!}h^2+\ldots,$$

which is Taylor's Series.

Ex. 1. From (7) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + R_n$, where R_n (Lagrange's form) $= \sin\left(\theta x + n\frac{\pi}{2}\right)\frac{x^n}{n!}$. Now $\frac{x^n}{n!} = \frac{x}{1}\cdot\frac{x}{2}\cdot\frac{x}{3}\ldots$, and each fraction after a certain point is numerically <1, hence the limit of the product $=0$ for $n=\infty$. Also $\sin\left(\theta x + n\frac{\pi}{2}\right)$ remains finite as n increases, since it cannot be >1 or <-1. Thus $\mathcal{L}_{n=\infty} R_n = 0$ for all values of x. Hence, for all values of x,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Similarly, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ for all values of x.

2. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + R_n$, where R_n (Cauchy's form)

$$= \frac{(-1)^{n-1}(1-\theta)^{n-1}x^n}{(1+\theta x)^n} = \frac{(-1)^{n-1}}{1-\theta}\left(\frac{1-\theta}{\frac{1}{x}+\theta}\right)^n.$$

But $1-\theta < \left|\frac{1}{x}+\theta\right|$ if $-1 < x \leqq 1$. Also $1-\theta$ remains finite. Hence, for these values of x,

$$\mathcal{L}_{n=\infty} R_n = 0, \quad \text{and} \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} -$$

For all other values of x the series is non-convergent (§196) and hence cannot represent the function.

3. Show that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ for all values of x.

CHAPTER XL.

FOURIER'S SERIES.

211. (*a*) A function $f(x)$ may be developed into an infinite series of the form

$$A+a_1\cos x+a_2\cos 2x+\ldots+a_n\cos nx+\ldots, \quad (1)$$

consisting of a constant term A, and cosines of x and multiples of x, with constant coefficients $a_1, a_2, \ldots$ For any value of x from $x=0$ to $x=\pi$ the series will represent the function, i.e., the limit of the sum of the terms of the series will be equal to the value of the function.

(*b*) $f(x)$ may also be developed into a sine series of the form

$$a_1\sin x+a_2\sin 2x+\ldots+a_n\sin nx+\ldots \quad (2)$$

and the series will represent the function for any value of x *between* 0 and π.

(*c*) $f(x)$ may be developed into a sine and cosine series of the form

$$A+a_1\sin x+a_2\sin 2x+\ldots+b_1\cos x+b_2\cos 2x+\ldots, \quad (3)$$

and the series will represent the function for any value of x *between* $-\pi$ and π.

Whether the series represent the function for values of x other than those stated will depend upon the nature of the function.

The above propositions were fully investigated for the first time by Fourier (Théorie analytique de la Chaleur, 1882) and the series are called Fourier's Series.

212. Assuming the form of the series we shall explain a method of calculating the constants. It will be necessary to make use of the following results of integration which may be easily verified by the student (see Ch. XXII).

Supposing n and m to be integers and $n \neq m$, the following integrals have the values herewith given:

	From 0 to π.	From $-\pi$ to π.
$\int \cos nx\, dx$	0	0
$\int \sin nx\, dx$		0
$\int \cos mx \cos nx\, dx$	0	0
$\int \sin mx \sin nx\, dx$	0	0
$\int \sin mx \cos nx\, dx$		0
$\int \cos^2 nx\, dx$	$\frac{1}{2}\pi$	π
$\int \sin^2 nx\, dx$	$\frac{1}{2}\pi$	π

213. The cosine series. Suppose

$$f(x) = A + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx + \ldots \quad (1)$$

An operation may be performed which causes every term of the series to disappear except that of which the constant is desired. Multiply (1) by dx and integrate between 0 and π. Then

$$\int_0^\pi f(x)dx = A\pi, \quad \therefore\ A = \frac{1}{\pi}\int_0^\pi f(x)\, dx.$$

Multiply (1) by $\cos nx\, dx$ and integrate between 0 and π. Then

$$\int_0^\pi f(x) \cos nx\, dx = a_n \cdot \tfrac{1}{2}\pi, \quad \therefore\ a_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx\, dx.$$

Ex. 1. Let $f(x)=x$. Then $A=\frac{1}{\pi}\int_0^{\pi} x\,dx=\frac{\pi}{2}$.

Also, $a_n=\frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx=-\frac{2}{\pi n^2}(1-\cos n\pi)$.

Hence, making $n=1, 2, 3, \ldots$,

$$x=\frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\ldots\right).$$

In Fig. 124, I, II, III represent the graphs of the first three terms from $x=0$ to $x=\pi$, and AB that of their sum. The limit of AB for the infinite series is the straight line OC, or $y=x$.

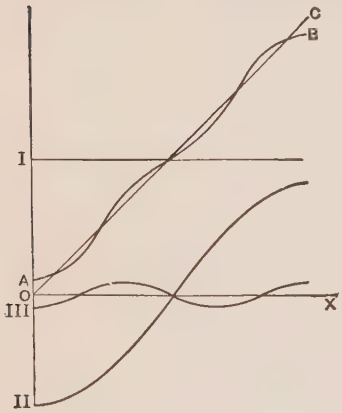


Fig. 124.

The series holds from $x=0$ to $x=\pi$; for smaller or greater values of x the series does not represent x. The value of each term of the series is unchanged when the sign of x is changed, and is repeated whenever x changes by the amount 2π. Hence the graph of the series consists of the lines of Fig. 125 continued indefinitely in both directions, or the equation of all these lines is

$$y=\frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{1}{3^2}\cos 3x+\ldots\right).$$

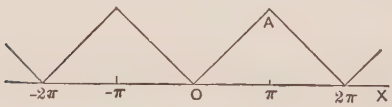


Fig. 125.

The axes may be transferred to any other position in the usual way. Thus to make the middle point of OA the origin, the x-axis

being parallel to that of the figure, change y into $y+\frac{1}{2}\pi$, and x into $x+\frac{1}{2}\pi$. The result is

$$y=\frac{4}{\pi}\left(\sin x-\frac{1}{3^2}\sin 3x+\ldots\right).$$

Since $mx=\dfrac{m\pi}{2}-\dfrac{4m}{\pi}\left(\cos x+\dfrac{1}{3^2}\cos 3x+\ldots\right)$, the terms on the right represent any line $y=mx$ through the origin, x varying from 0 to π, and the equation of all lines like those of Fig. 125 making angles $\pm\tan^{-1}m$ with the x-axis is

$$y=\frac{m\pi}{2}-\frac{4m}{\pi}\left(\cos x+\frac{1}{3^2}\cos 3x+\ldots\right).$$

2. Making $x=0$ in Ex 1, show that $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots=\dfrac{\pi^2}{8}$, and hence that $\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\ldots=\dfrac{\pi^2}{6}$, and $\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\ldots=\dfrac{\pi^2}{12}$.

3. $x^2=\dfrac{\pi^2}{3}-4\left(\cos x-\dfrac{1}{2^2}\cos 2x+\dfrac{1}{3^2}\cos 3x-\ldots\right)$, for $[-\pi,\ \pi]$.*

In this, as in all other cases, the cosine series of an even function (see Ex. 12, p. 40) represents the function for negative values of x as well as for positive values. The graph consists of a series of parabolas of breadth 2π and height π^2 (Fig. 126).

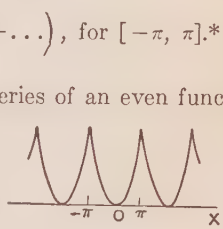


FIG. 126.

4. $\sin x=\dfrac{2}{\pi}-\dfrac{4}{\pi}\left(\dfrac{\cos 2x}{1\,.\,3}+\dfrac{\cos 4x}{3\,.\,5}+\ldots\right)$, for $[0,\ \pi]$.

5. $\cosh x=\dfrac{\sinh\pi}{\pi}\left[1-2\left(\dfrac{\cos x}{1+1^2}-\dfrac{\cos 2x}{1+2^2}+\ldots\right)\right]$, for $[-\pi,\ \pi]$, and hence

$$\frac{1}{1+1^2}+\frac{1}{1+2^2}+\frac{1}{1+3^2}+\ldots=\tfrac{1}{2}(\pi\coth\pi-1),$$

$$\frac{1}{1+1^2}-\frac{1}{1+2^2}+\frac{1}{1+3^2}-\ldots=\tfrac{1}{2}(1-\pi\operatorname{cosech}\pi).$$

* See § 22.

214. The sine series. Each of the cosine expansions gives on differentiation a sine expansion of the form

$$f(x) = a_1 \sin x + a_2 \sin 2x + \ldots + a_n \sin nx + \ldots \qquad (1)$$

The series may be obtained without reference to the cosine series as follows: Multiply (1) by $\sin nx\,dx$ and integrate between 0 and π. Then

$$\int_0^\pi f(x) \sin nx\,dx = a_n \cdot \tfrac{1}{2}\pi, \therefore\ a_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\,dx.$$

Ex. 1. Let $f(x) = 1$. Here $a_n = \dfrac{2}{n\pi}(1 - \cos n\pi)$.

Hence $$1 = \frac{4}{\pi}(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \ldots).$$

Fig. 127 represents the graphs of the first three terms and of their sum. The limit of the latter graph is $O'C$, the line $y = 1$. The limit of the sum of the series is 1 for all values of x *between* 0 and π. Thus the limit of the y of the graph of the series for $x = 0$ or π from inside the interval is 1, but its *value* for $x = 0$ or π is 0. The series

$$\frac{4h}{\pi}(\sin x + \tfrac{1}{3}\sin 3x + \ldots)$$

represents any constant h, the amplitudes of the sine curves being $4h/\pi$, $4h/3\pi$, ... The complete graph of the series con-

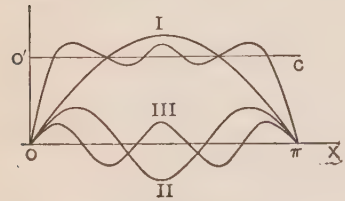


Fig. 127.

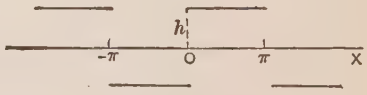


Fig. 128.

sists of the straight lines of Fig. 128 continued indefinitely in both directions, and the equation of all these lines is

$$y = \frac{4h}{\pi}(\sin x + \tfrac{1}{3}\sin 3x + \ldots).$$

Although the series is convergent, the series formed by the derivatives of its terms is non-convergent, and therefore does not represent the slope of the graph (the derivative of the function) at any point.

2. $x = 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots)$.

This represents x for $0 < x < \pi$. Since the series changes sign with x, and the function is an odd one, the series represents the function for negative as well as positive values of x. Hence it holds for $-\pi < x < \pi$. The graph consists of a series of straight lines, as in Fig. 129.

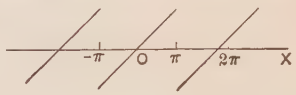


FIG. 129.

3. $\cos x = \frac{4}{\pi}\left(\frac{2\sin 2x}{1\,.\,3} + \frac{4\sin 4x}{3\,.\,5} + \dots\right)$, for $]0, \pi[$.

Draw the graph.

4. $x^2 = \frac{2}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x + \dots - \frac{\pi^2}{2}\sin 2x - \frac{\pi^2}{4}\sin 4x - \dots\right]$, for $[0, \pi[$.

5. From Ex. 4 show that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}$.

6. Show that

$$\frac{\pi}{4} = \sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \dots, \quad \text{for} \quad]0, \pi[,$$

$$\frac{\pi}{4} = \cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \dots, \quad \text{for} \quad \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

215. The function represented by a Fourier series need not be a single function throughout the range π of the value of x; the same series may represent one function for a part of the range and one or more other functions for the remainder of the range.

Let $$f_1(x) = A + a_1\cos x + a_2\cos 2x + \dots \qquad (1)$$

for $x = 0$ to $x = \alpha$, and

$$f_2(x) = A + a_1\cos x + a_2\cos 2x + \dots \qquad (2)$$

(the same series) for $x=\alpha$ to $x=\pi$. Multiply (1) by dx and integrate between 0 and α, also multiply (2) by dx and integrate between α and π, and add the results. Then each term of the series is, on the whole, integrated between 0 and π. Hence

$$\int_0^\alpha f_1(x)\,dx+\int_\alpha^\pi f_2(x)\,dx=A\pi,$$

$$\therefore\ A=\frac{1}{\pi}\Big[\int_0^\alpha f_1(x)\,dx+\int_\alpha^\pi f_2(x)\,dx\Big].$$

Similarly, $a_n=\frac{2}{\pi}\Big[\int_0^\alpha f_1(x)\cos nx\,dx+\int_\alpha^\pi f_2(x)\cos nx\,dx\Big]$

for the cosine series, and

$$a_n=\frac{2}{\pi}\Big[\int_0^\alpha f_1(x)\sin nx\,dx+\int_\alpha^\pi f_2(x)\sin nx\,dx\Big]$$

for the sine series.

The series may not hold at the point or points where the change of function occurs.

It may be noticed that A in the cosine series is always equal to the mean height of the graph from $x=0$ to $x=\pi$.

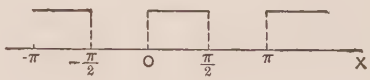


Fig. 130.*

Ex. 1. To find a cosine series which $=1$ for $0<x<\frac{1}{2}\pi$, and $=0$ for $\frac{1}{2}\pi<x<\pi$.

$$A=\frac{1}{2},\quad\text{and}\quad a_n=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos nx\,dx=\frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

Hence the series is

$$\frac{1}{2}+\frac{2}{\pi}(\cos x-\tfrac{1}{3}\cos 3x+\tfrac{1}{5}\cos 5x-\dots).$$

* The electrician's make-and-break curve.

It is true when $x=0$ and $x=\pi$, but $=\frac{1}{2}$ when $x=\frac{1}{2}\pi$.

2. Find a sine series for the same.

$$Ans.\ \frac{2}{\pi}\left(\frac{\sin x}{1}+\frac{2\sin 2x}{2}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\frac{2\sin 6x}{6}+\frac{\sin 7x}{7}+\dots\right)$$
$$=\frac{1}{2}+\frac{2}{\pi}(\sin 2x+\tfrac{1}{3}\sin 6x+\tfrac{1}{5}\sin 10x+\dots).$$

(See § 214, Ex. 6.)

216. The cosine and sine series. For values of x between $-\pi$ and π,

$$f(x)=A+a_1\cos x+\dots+a_n\cos nx+\dots+b_1\sin x+\dots+b_n\sin nx+\dots$$

It is easily shown, as in § 213, that the constants may be determined as follows:

$$A=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx,$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx.$$

Ex. $$e^x=\frac{2\sinh\pi}{\pi}\left(\frac{1}{2}+\dots+\frac{\cos n\pi}{n^2+1}\cos nx+\dots-\frac{n\cos n\pi}{n^2+1}\sin nx-\dots\right).$$

217. By the following method a cosine series which will hold for values of x from 0 to any number c (instead of π) may be obtained. If $x=cz/\pi$, $x=0$ when $z=0$, and $x=c$ when $z=\pi$. Hence, in $f(x)$ change x into cz/π, develop in terms of z, and change z into $\pi x/c$. Similarly to obtain a sine series for values of x between 0 and c, or a cosine and sine series for values of x between $-c$ and c. In all cases the constant term is equal to the mean height of the graph (or the mean value of the function) for the interval in ques-

tion. In this way the series already obtained may be adapted to the intervals stated below.

Ex. 1. $x = \frac{c}{2} - \frac{4c}{\pi^2}\left(\cos\frac{\pi x}{c} + \frac{1}{3^2}\cos\frac{3\pi x}{c} + \ldots\right)$, $[0, c]$.

2. $x^2 = \frac{c^2}{3} - \frac{4c^2}{\pi^2}\left(\cos\frac{\pi x}{c} - \frac{1}{2^2}\cos\frac{2\pi x}{c} + \ldots\right)$, $[-c, c]$.

3. $h = \frac{4h}{\pi}\left(\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \ldots\right)$, $]0, c[$.

4. $x = \frac{2c}{\pi}\left(\sin\frac{\pi x}{c} - \frac{1}{2}\sin\frac{2\pi x}{c} + \ldots\right)$, $]-c, c[$.

5. $\frac{\pi}{4} = \sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \ldots$, $]0, c[$.

218. If a function of x is developed into a series for the interval $-c$ to c, and if the values of the function are repeated periodically for every interval $2c$ of x, the series will continue to represent those values as x increases or decreases. In other words, the periodic function of period $2c$ is developed into a series consisting of a constant term and harmonic functions of periods $2c$, $2c/2$, $2c/3$, etc. Fourier's Theorem is to the effect that this development is always possible, the complete series being of the form

$$A + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \ldots + b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \ldots,$$

which is equivalent to

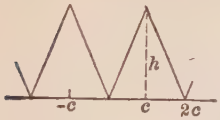
$$A + A_1\sin\left(\frac{\pi x}{c} + \alpha_1\right) + A_2\sin\left(\frac{2\pi x}{c} + \alpha_2\right) + \ldots,$$

where $A_n = \sqrt{a_n^2 + b_n^2}$ and $\alpha_n = \tan^{-1}(a_n/b_n)$.

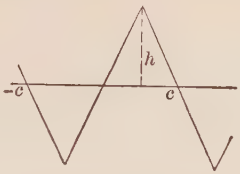
As already stated, the function may consist of distinct functions for parts of the interval.

EXAMPLES.

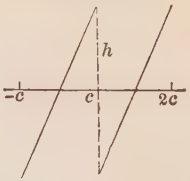
Develop the functions represented by the following figures:



A



B

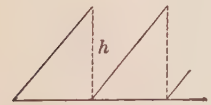


C

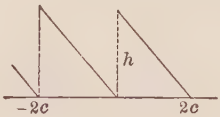
1. Fig. A. *Ans.* $\frac{h}{2}-\frac{4h}{\pi^2}\left(\cos\frac{\pi x}{c}+\frac{1}{3^2}\cos\frac{3\pi x}{c}+\ldots\right)$.

2. Fig. B. $\frac{8h}{\pi^2}\left(\sin\frac{\pi x}{c}-\frac{1}{3^2}\sin\frac{3\pi x}{c}+\ldots\right)$.

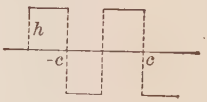
3. Fig. C. $\frac{2h}{\pi}\left(\sin\frac{\pi x}{c}-\frac{1}{2}\sin\frac{2\pi x}{c}+\ldots\right)$.



D



E

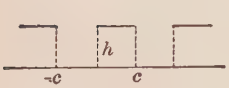


F

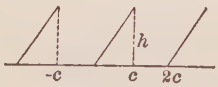
4. Fig. D. $\frac{h}{2}-\frac{h}{\pi}\left(\sin\frac{\pi x}{c}+\frac{1}{2}\sin\frac{2\pi x}{c}+\ldots\right)$.

5. Fig. E. $\frac{h}{2}+\frac{h}{\pi}\left(\sin\frac{\pi x}{c}+\frac{1}{2}\sin\frac{2\pi x}{c}+\ldots\right)$.

6. Fig. F. $\frac{4h}{\pi}\left(\sin\frac{\pi x}{c}+\frac{1}{3}\sin\frac{3\pi x}{c}+\ldots\right)$.



G



H

J

7. Fig. G. $\frac{h}{2}+\frac{2h}{\pi}\left(\sin\frac{\pi x}{c}+\frac{1}{3}\sin\frac{3\pi x}{c}+\ldots\right)$.

8. Fig. H. $\frac{h}{4}-\frac{2h}{\pi^2}\left(\cos\frac{\pi x}{c}+\frac{1}{3^2}\cos\frac{3\pi x}{c}+\ldots\right)$.

$$+\frac{h}{\pi}\left(\sin\frac{\pi x}{c}-\frac{1}{2}\sin\frac{2\pi x}{c}+\ldots\right).$$

9. Fig. J. (Parabolas. Latus rectum $=2c$).

$$\frac{c}{3}-\frac{2c}{\pi^2}\left(\cos\frac{\pi x}{c}+\frac{1}{2^2}\cos\frac{2\pi x}{c}+\ldots\right).$$

10. The displacements of a slide-valve actuated by a Gooch link were measured at eight intervals each of 45°, and found to be as follows, beginning with the crank on the inner dead-centre:

$$2{\cdot}44,\quad 1{\cdot}65,\quad 0,\quad -1{\cdot}37,\quad -1{\cdot}87,\quad -1{\cdot}37,\quad 0,\quad 1{\cdot}65.$$

Assuming that the motion of the valve is compounded of two simple harmonic motions, one of double the frequency of the other, as represented by the equation

$$y=k+a\sin(\theta+\alpha)+b\sin(2\theta+\beta),$$

where θ is the crank angle, find the values of k, a, α, b, β. (Castle, *Manual of Practical Mathematics.*)

There are various graphical or other practical methods by which the coefficients of a small number of terms of a Fourier series may be found, but in this example an algebraical solution will suffice. Assume

$$y=k+a_1\sin\theta+b_1\cos\theta+a_2\sin 2\theta+b_2\cos 2\theta,$$

substitute the given values of y for $\theta=0$, 45°, 90°, etc., and solve the equations.

Ans. $y={\cdot}14+2{\cdot}16\cos\theta+{\cdot}14\cos 2\theta,$

or $\quad={\cdot}14-2{\cdot}16\sin(\theta+90°)+{\cdot}14\sin(2\theta+90°).$

CHAPTER XLI.

APPROXIMATE INTEGRATION. ELLIPTIC INTEGRALS.

219. Approximate integration. If the general value of $\int f(x)\,dx$ cannot be obtained it may be possible to find a sufficiently close approximation to the desired result.

(1) If $f(x)$ can be developed into a rapidly converging series, the integration of a few terms will give an approximate value of the integral.

(2) The curve $y=f(x)$ may be plotted when $f(x)$ is given. Its area obtained by Simpson's Rule (§ 131) or by the planimeter (Appendix, Note D) will give an approximate value of $\int f(x)\,dx$ between assigned values of x.

(3) $\int y^2\,dx$ and $\int y^3\,dx$ as well as $\int y\,dx$ for a curve which has been drawn mechanically or otherwise can be obtained mechanically. The result, although theoretically exact, is affected by observation and instrumental error. On Mechanical Integration see Appendix, Note D.

Elliptic Integrals.

220. $$\int \frac{d\theta}{\sqrt{1-m^2\sin^2\theta}}, \quad \int \sqrt{1-m^2\sin^2\theta}\,d\theta,$$

and $$\int \frac{d\theta}{(1+a\sin^2\theta)\sqrt{1-m^2\sin^2\theta}}$$

are called *elliptic integrals* of the first, second, and third class respectively. The constant m, which is assumed to be not greater than unity, is called the *modulus* of the integrals. The lower limit is understood to be 0 in each case, and, the angle varying from 0 to θ, θ is called the *amplitude* of the integral. The integrals are represented by the symbols $F(m, \theta)$, $E(m, \theta)$, and $\Pi(a, m, \theta)$, respectively; or by $F_m(\theta)$, etc. When the limits are 0 and $\frac{1}{2}\pi$ (i.e., when the amplitude is $\frac{1}{2}\pi$) the integrals are said to be *complete*.

If $\sin\theta = x$, the integrals become

$$\int \frac{dx}{\sqrt{(1-x^2)(1-m^2x^2)}}, \quad \int \sqrt{\frac{1-m^2x^2}{1-x^2}}\, dx,$$

and

$$\int \frac{dx}{(1+ax^2)\sqrt{(1-x^2)(1-m^2x^2)}},$$

and they are complete when the limits are 0 and 1.

221. The values of the elliptic integrals cannot be expressed in finite terms, but may be approximated to by infinite series.

Thus by the Binomial Theorem

$$\frac{d\theta}{\sqrt{1-m^2\sin^2\theta}} = (1-m^2\sin^2\theta)^{-\frac{1}{2}}\, d\theta$$

$$= \left(1+\frac{1}{2}m^2\sin^2\theta+\frac{1\,.\,3}{2\,.\,4}m^4\sin^4\theta+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}m^6\sin^6\theta+\ldots\right)d\theta,$$

and each term may be integrated by § 113 (see Ex. 12 below).

Taking the limits as 0 and $\frac{1}{2}\pi$ we have (§ 120) for the complete elliptic integral of the first class

$$F(m, \tfrac{1}{2}\pi) = \frac{\pi}{2}\left[1+\left(\frac{1}{2}m\right)^2+\left(\frac{1\,.\,3}{2\,.\,4}m^2\right)^2+\left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}m^3\right)^2+\ldots\right].$$

Similarly for the integral of the second class we have

$$\sqrt{1-m^2\sin^2\theta}\,d\theta=(1-m^2\sin^2\theta)^{\frac{1}{2}}\,d\theta$$

$$=\left(1-\frac{1}{2}m^2\sin^2\theta-\frac{1}{2\,.\,4}m^4\sin^4\theta-\frac{1\,.\,3}{2\,.\,4\,.\,6}m^6\sin^6\theta-\ldots\right)d\theta,$$

and

$$\frac{\pi}{2}\left[1-\left(\frac{1}{2}m\right)^2-\frac{1}{3}\left(\frac{1\,.\,3}{2\,.\,4}m^2\right)^2-\frac{1}{5}\left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}m^3\right)^2-\ldots\right]$$

for the complete integral of the second class, $E(m, \frac{1}{2}\pi)$.

Three-figure tables of the integrals for certain values of the modulus and amplitude are given at the end of this volume.

It may be noticed that

$$E(0,\theta)=F(0,\theta)=\theta \text{ (in radians)};$$

also,
$$E(1,\theta)=\int\cos\theta\,d\theta=\sin\theta,$$

$$F(1,\theta)=\int\frac{d\theta}{\cos\theta}=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\lambda(\theta).$$

222. From the above expansions and the integral (§ 113) of $\sin^n\theta\,d\theta$ it may be shown that

$$E(m, n\pi\pm\theta)=2nE\pm E(m,\theta),$$
$$F(m, n\pi\pm\theta)=2nK\pm F(m,\theta),$$

E and K being the values of the integrals for the amplitude $\frac{1}{2}\pi$, and n being any integer. Hence a table of the elliptic integrals in which the amplitude varies from 0 to $\frac{1}{2}\pi$ may be used for all higher values of the amplitude.

EXAMPLES.

1. To find the length of an arc of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The complement of the eccentric angle being denoted by θ we have $x=a\sin\theta$, and $y=b\cos\theta$.

$$\therefore\ dx=a\cos\theta\,d\theta,\quad dy=-b\sin\theta\,d\theta$$

whence $ds^2 = dx^2 + dy^2 = (a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta^2$
$= [a^2 - (a^2 - b^2)\sin^2\theta]\,d\theta^2 = a^2(1 - m^2\sin^2\theta)\,d\theta^2,$

where m is the eccentricity of the ellipse. Hence the length of the elliptic arc measured from the end of the minor axis is

$$a\int_0^{\theta_1}\sqrt{1-m^2\sin^2\theta}\,d\theta = aE(m, \theta_1),$$

an elliptic integral of the second class. The length of the quadrant of the ellipse $= aE(m, \frac{1}{2}\pi)$.

2. Find the circumference of the ellipse $x^2 + 2y^2 = 2$.

Ans. 7·64.

3. Of the ellipse $3x^2 + 4y^2 = 12$ find (1) the length of the arc from $x = 0$ to $x = 1$, (2) the length of the quadrant, (3) the middle point of the quadrant. *Ans.* 1·036, 2·934, (1·36, 1·27).

4. An arc of the lemniscate $r^2 = a^2\cos 2\theta$.

From $ds^2 = r^2d\theta^2 + dr^2$ we have

$$s = \int_0^{\theta_1}\frac{d\theta}{\sqrt{1-2\sin^2\theta}}.$$

Let $2\sin^2\theta = \sin^2\phi$. Then

$$s = \frac{a}{\sqrt{2}}\int_0^{\phi_1}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}} = \frac{a}{\sqrt{2}}F\left(\frac{1}{\sqrt{2}}, \phi_1\right),$$

an elliptic integral of the first class. The length of a quadrant of the lemniscate is therefore

$$\frac{a}{\sqrt{2}}F\left(\frac{1}{\sqrt{2}}, \tfrac{1}{2}\pi\right) = 1{\cdot}311a.$$

If $\theta = 30°$, show that $s = {\cdot}584a$.

5. $\displaystyle\int_0^c\frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a}F\left(\frac{b}{a}, \sin^{-1}\frac{c}{b}\right)$. Let $x = b\sin\theta$.

6. $\displaystyle\int_0^c\frac{dx}{\sqrt{(a^2+x^2)(b^2+x^2)}} = \frac{1}{a}F\left(\frac{\sqrt{a^2-b^2}}{a}, \tan^{-1}\frac{c}{b}\right)$.

7. $\displaystyle\int_c^b\frac{dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}F\left(\frac{b}{\sqrt{a^2+b^2}}, \cos^{-1}\frac{c}{b}\right)$.

8. $\int_0^c \frac{dx}{\sqrt{x(1-x)(1-m^2x)}} = 2F(m, \sin^{-1}\sqrt{c})$.

9 A simple pendulum of length l oscillates through an angle β on each side of the vertical. To find the time of an oscillation.

When the pendulum makes an angle ϕ with the vertical, the acceleration $-g \sin \phi$ in the direction of the motion $= d^2s/dt^2 = l\, d^2\phi/dt^2$.

$$\therefore \frac{d^2\phi}{dt^2} = -\frac{g}{l}\sin \phi.$$

Multiply by $2\, d\phi$ and integrate. Then

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{2g}{l}(\cos \phi - \cos \beta) = \frac{4g}{l}(\sin^2 \tfrac{1}{2}\beta - \sin^2 \tfrac{1}{2}\phi).$$

Hence solving for dt and integrating,

$$\frac{1}{2}\sqrt{\frac{l}{g}}\int_0^\beta \frac{d\phi}{\sqrt{\sin^2 \tfrac{1}{2}\beta - \sin^2 \tfrac{1}{2}\phi}} \qquad (1)$$

is the time of a half oscillation. Let $\sin \tfrac{1}{2}\phi = \sin \tfrac{1}{2}\beta \sin \theta$. Then (1) becomes

$$\sqrt{\frac{l}{g}}\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1 - \sin^2 \tfrac{1}{2}\beta \sin^2 \theta}} = \sqrt{\frac{l}{g}}F(\sin \tfrac{1}{2}\beta, \tfrac{1}{2}\pi).$$

Hence the time of an oscillation is

$$2\sqrt{\frac{l}{g}}F(\sin \tfrac{1}{2}\beta, \tfrac{1}{2}\pi).$$

10. Find the time of oscillation of a pendulum when $\beta = 60°$.

Ans. $3{\cdot}372\sqrt{l/g}$.

Find the time through the lower half of the motion.

$$Ans.\ 2\sqrt{\frac{l}{g}}F\left(\sin 30°, \sin^{-1}\frac{\sin 15°}{\sin 30°}\right) = 1{\cdot}102\sqrt{l/g}.$$

11. If the arc s is small compared with the length l, show that the time of oscillation of a simple pendulum is approximately

$$\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{64}\frac{s^2}{l^2}\right).$$

12. Show that

$$F(m, \theta) = \theta + \tfrac{1}{4}m^2(\theta - \sin \theta \cos \theta) + \tfrac{3}{64}m^4(3\theta - 3 \sin \theta \cos \theta - 2 \sin^3\theta \cos \theta) + \ldots,$$

$$E(m, \theta) = \theta - \tfrac{1}{4}m^2(\theta - \sin \theta \cos \theta) - \tfrac{1}{64}m^4(3\theta - 3 \sin \theta \cos \theta - 2 \sin^3\theta \cos \theta) - \ldots$$

CHAPTER XLII.

SINGULAR FORMS.

223. We have already seen that for a certain value of the variable a function may assume the form $0/0$. The form is said to be *singular;* it is also called an *indeterminate* form.

There are other singular or indeterminate forms, such as ∞/∞, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞.

A function in a singular form has no value. Our object is to find the *limit* of the value of the function as the variable approaches the critical value in question.

224. The form o/o. The fraction $(x-1)/(x^3-1)$ takes the form $0/0$ when $x=1$. But

$$\frac{x-1}{x^3-1}=\frac{x-1}{(x-1)(x^2+x+1)}=\frac{1}{x^2+x+1},$$

provided that $x-1\neq 0$. The last fraction $\doteq \frac{1}{3}$ when $x\doteq 1$, hence the given fraction $\doteq \frac{1}{3}$ when $x\doteq 1$.

Method of the Calculus. Let the fraction be $f(x)/F(x)$, and suppose a to be the value of x which causes the fraction to take the form $0/0$, or that $f(a)=0$ and $F(a)=0$; also that $x=a+h$, where h is a small quantity which is to $\doteq 0$. Then, assuming the functions to be such that the expansion of Taylor's Theorem applies,

$$\frac{f(a+h)}{F(a+h)}=\frac{f(a)+f'(a)h+\frac{f''(a)}{1\cdot 2}h^2+\dots}{F(a)+F'(a)h+\frac{F''(a)}{1\cdot 2}h^2+\dots}=\frac{f'(a)+\frac{f''(a)}{1\cdot 2}h+\dots}{F'(a)+\frac{F''(a)}{1\cdot 2}h+\dots}.$$

Hence when $h \doteq 0$, i.e., when $x \doteq a$, the given fraction $\doteq f'(a)/F'(a)$, or

$$\mathcal{L}\frac{f(x)}{F(x)} = \frac{f'(a)}{F'(a)}.$$

If $f'(a)$ and $F'(a)$ are also 0, it may be shown in the same way that $\mathcal{L}\dfrac{f(x)}{F(x)} = \dfrac{f''(a)}{F''(a)}$, and so on.

Ex. 1. If $\dfrac{f(x)}{F(x)} = \dfrac{x-1}{x^3-1}$, $\dfrac{f'(x)}{F'(x)} = \dfrac{1}{3x^2} = \dfrac{1}{3}$ when $x = 1$.

$$\therefore \mathcal{L}\frac{x-1}{x^3-1} = \frac{1}{3} \text{ when } x \doteq 1.$$

The work may be conveniently expressed thus: When $x \doteq 1$,

$$\mathcal{L}\frac{x-1}{x^3-1} = \frac{1}{3x^2}\Big]_1 = \frac{1}{3}.$$

2. If $x \doteq 0$, $\mathcal{L}\dfrac{e^x - e^{-x}}{\sin x} = \dfrac{e^x + e^{-x}}{\cos x}\Big]_0 = \dfrac{2}{1} = 2.$

3. $\dfrac{f(x)}{F(x)} = \dfrac{e^x - 1 - \log(1+x)}{x^2} = \dfrac{0}{0}$ when $x = 0$.

$$\frac{f'(x)}{F'(x)} = \frac{e^x - \dfrac{1}{1+x}}{2x} = \frac{0}{0} \text{ when } x = 0.$$

$$\frac{f''(x)}{F''(x)} = \frac{e^x + \dfrac{1}{(1+x)^2}}{2} = 1 \text{ when } x = 0.$$

$$\therefore \mathcal{L}\frac{e^x - 1 - \log(1+x)}{x^2} = 1 \text{ when } x \doteq 0.$$

225. The form ∞/∞. Let $f(x)/F(x) = \infty/\infty$, first when $x = \infty$. Let the graphs of $f(x)$ and $F(x)$ be PQ and $P'Q'$, Fig. 131, and let $OM = x$. Let the limits of the tangents at

P and P' be the asymptotes AS and $A'S'$ when $x=\infty$, and let $A'A=c$. Then $MP=f(x)$, $MP'=F(x)$, $\tan MTP=f'(x)$,

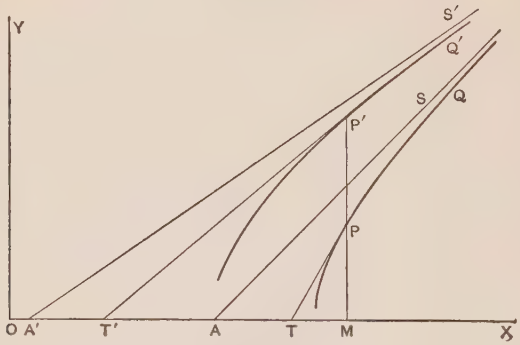


Fig. 131.

$\tan MT'P'=F'(x)$. Hence

$$\frac{f(x)}{F(x)}=\frac{TM}{T'M}\frac{f'(x)}{F'(x)}.$$

But $$\pounds\frac{TM}{T'M}=\pounds\frac{AM}{A'M}=\pounds\left(1-\frac{c}{A'M}\right)=1$$

when $A'M=\infty$. Hence

$$\pounds\frac{f(x)}{F(x)}=\pounds\frac{f'(x)}{F'(x)}. \qquad (1)$$

Secondly, let $f(x)/F(x)=\infty/\infty$ when $x\doteq a$. For x substitute $a+1/z$. Then $z=\infty$ when $x\doteq a$. But by (1), if $z=\infty$,

$$\pounds\frac{f\left(a+\frac{1}{z}\right)}{F\left(a+\frac{1}{z}\right)}=\pounds\frac{f'\left(a+\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)}{F'\left(a+\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)}=\pounds\frac{f'\left(a+\frac{1}{z}\right)}{F'\left(a+\frac{1}{z}\right)},$$

or $$\pounds\frac{f(x)}{F(x)}=\pounds\frac{f'(x)}{F'(x)}.$$

Hence the result (1) holds in this case also. Thus when a fraction has the form ∞/∞ the limit of its value is found from the same differentiations as when it has the form $0/0$.

Ex. $$\frac{f(x)}{F(x)}=\frac{\log\cos\frac{\pi x}{2}}{\log(1-x)}=\frac{\infty}{\infty} \text{ when } x=1.$$

$$\frac{f'(x)}{F'(x)}=\frac{-\frac{\pi}{2}\tan\frac{\pi x}{2}}{-\frac{1}{1-x}}=\frac{\pi}{2}\cdot\frac{1-x}{\cot\frac{\pi x}{2}}=\frac{0}{0} \text{ when } x=1.$$

But (§ 224) $$\frac{\pi}{2}\cdot\frac{1-x}{\cot\frac{\pi x}{2}}=\frac{\pi}{2}\cdot\frac{-1}{-\frac{\pi}{2}\operatorname{cosec}^2\frac{\pi x}{2}}\Bigg]_1=1.$$

Hence the given fraction $=1$ when $x=1$.

226. The forms $0.\infty$, $\infty-\infty$. A function which assumes the form $0.\infty$, or $\infty-\infty$ may, by an algebraical or other change, be made to take the form $0/0$ or ∞/∞.

Ex. 1. $x(1-e^{-\frac{a}{x}})$ tends to $\infty\,.\,0$ when $x=\infty$. The limit is most easily found by using the exponential series.

For $$x(1-e^{-\frac{a}{x}})=x\left[1-\left(1-\frac{a}{x}+\frac{a^2}{2x^2}-\ldots\right)\right]$$
$$=a-\frac{a^2}{2x}+\ldots=a \text{ when } x=\infty.$$

2. $(1-x)\tan\frac{\pi x}{2}$ tends to the form $0\,.\,\infty$ when $x=1$.

But it $=\dfrac{1-x}{\cot\frac{\pi x}{2}}$, which $=\dfrac{0}{0}$ when $x=1$.

$\therefore$ (§ 224) we have $\dfrac{-1}{-\frac{\pi}{2}\operatorname{cosec}^2\frac{\pi x}{2}}=\dfrac{2}{\pi}$ when $x=1$.

$$\therefore \mathcal{L}(1-x)\tan\frac{\pi x}{2}=\frac{2}{\pi} \text{ when } x \doteq 1.$$

3. $\dfrac{x}{x-1}-\dfrac{1}{\log x}=\infty-\infty$ when $x \doteq 1$.

But $\dfrac{x}{x-1}-\dfrac{1}{\log x}=\dfrac{x\log x-x+1}{(x-1)\log x}=\dfrac{0}{0}$ when $x=1$,

whence (§ 224) $\dfrac{\log x}{1-\dfrac{1}{x}+\log x}$, which $=\dfrac{0}{0}$ when $x=1$.

A second differentiation gives $\dfrac{\dfrac{1}{x}}{\dfrac{1}{x^2}+\dfrac{1}{x}}$, which $=\dfrac{1}{2}$ when $x=1$.

$\therefore$ ½ is the limit of $\dfrac{x}{x-1}-\dfrac{1}{\log x}$ when $x \doteq 1$.

227. The forms 0^0, ∞^0, 1^∞. Functions which assume these forms may be made to assume the form $0.\infty$ and therefore $0/0$ or ∞/∞ by first taking logarithms.

Ex. 1. $f(x)=x^{\frac{a}{\log\sin x}}$ tends to the form 0^0 when $x \doteq 0$.

But $\log f(x)=\dfrac{a}{\log\sin x}.\log x=a\dfrac{\log x}{\log\sin x}=\dfrac{\infty}{\infty}$ when $x \doteq 0$.

Differentiating (§ 225), $a\dfrac{\dfrac{1}{x}}{\dfrac{\cos x}{\sin x}}=a\dfrac{\tan x}{x} \doteq a$ when $x \doteq 0$.

$\therefore \mathcal{L}\log f(x)=a$. But * $\mathcal{L}\log f(x)=\log \mathcal{L}f(x)$, $\therefore \mathcal{L}f(x)=e^a$.

2. $f(x)=x^{\frac{1}{1-x}}$ tends to the form 1^∞ when $x \doteq 1$.

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* If v is any variable and $\mathcal{L}v=b$, $(b\neq 0)$, then (§ 8) $v=b+i=b(1+i/b)$.

$\therefore \log v-\log b=\log(1+i/b)$, which $\doteq 0$.

$\therefore \mathcal{L}\log v=\log b=\log \mathcal{L}v$.

But $\log f(x) = \dfrac{1}{1-x}\log x = \dfrac{\log x}{1-x} = \dfrac{0}{0}$ when $x = 1$.

$$\therefore \mathfrak{L}\log f(x) = \left.\frac{\frac{1}{x}}{-1}\right]_1 = -1, \quad \therefore x^{\frac{1}{1-x}} \doteq \frac{1}{e} \text{ when } x \doteq 1.$$

EXAMPLES.

1. When $x \doteq 0$ show that

(1) $\dfrac{\sin x}{x} \doteq 1$, (2) $\dfrac{\tan x}{x} \doteq 1$, (3) $\dfrac{e^x - e^{-x}}{\log(1+x)} \doteq 2$,

(4) $\dfrac{\cos x - \cos mx}{\cos x - \cos nx} \doteq \dfrac{1-m^2}{1-n^2}$, (5) $\dfrac{\log \sec x}{x^2} \doteq \dfrac{1}{2}$,

(6) $\dfrac{\log \sin 2x}{\log \sin x} \doteq 1$, (7) $x^n \log x \doteq 0$,

(8) $x^{\frac{a+x}{\log x}} \doteq e^a$, (9) $x^x \doteq 1$.

2. $\dfrac{x^3 - 3x + 2}{x^3 + 4x^2 - 5} \doteq 0$ when $x \doteq 1$.

3. If $x = \infty$, (1) $\dfrac{e^x}{x} = \infty$, (2) $2^x \sin \dfrac{a}{2^x} \doteq a$.

4. If $x \doteq \dfrac{\pi}{2}$, (1) $(\sin x)^{\sec 2x} \doteq \dfrac{1}{\sqrt{e}}$, (2) $\sec x\left(\dfrac{\pi}{2} - x \sin x\right) \doteq 1$

5. $(\sin x)^{\tan x} \doteq 1$ when $x \doteq 0$ or $\dfrac{\pi}{2}$.

6. $\left(1 + \dfrac{a}{x}\right)^x \doteq e^a$ when $x = \infty$, and $\doteq 1$ when $x \doteq 0$.

7. $\dfrac{\sec x}{\sec 3x} = \dfrac{\cos 3x}{\cos x} \doteq -3$ when $x \doteq \dfrac{\pi}{2}$.

8. $\dfrac{\tan x}{\tan 3x} = \dfrac{\cos 3x}{\cos x} \cdot \dfrac{\sin x}{\sin 3x} \doteq (-3)(-1) \doteq 3$ when $x \doteq \dfrac{\pi}{2}$.

9. $\dfrac{(e^x - 1)\tan^2 x}{x^3} = \left(\dfrac{e^x - 1}{x}\right)\left(\dfrac{\tan x}{x}\right)^2 \doteq 1$ when $x \doteq 0$.

CHAPTER XLIII.

SUCCESSIVE DIFFERENTIALS OF FUNCTIONS OF MORE THAN ONE VARIABLE. EXTENSION OF TAYLOR'S THEOREM. MAXIMA AND MINIMA FROM TAYLOR'S THEOREM.

Successive Partial Differentials.

228. Suppose u to be ax^3-xy^2+y. We have as in § 45, supposing x alone to vary,

$$d_xu=(3ax^2-y^2)dx,\ d_x^2u=6ax\,dx^2,\ d_x^3u=6a\,dx^3,\ d_x^4u=0,$$

$$d_yu=(-2xy+1)dy,\ d_y^2u=-2x\,dy^2,\ d_y^3u=0.$$

Again, d_xu or $(3ax^2-y^2)\,dx$ contains y as well as x, and we may obtain its differential on the supposition that y alone varies. We then have

$$d_yd_xu=-2y\,dy\,dx,\ d_y^2d_xu=-2\,dy^2dx,\ d_y^3d_xu=0.$$

Similarly, $d_xd_yu=-2y\,dx\,dy,\ d_xd_y^2u=-2\,dx\,dy^2,\ d_xd_y^3u=0.$

229. In comparing these results it will be seen that

$$d_xd_yu=d_yd_xu,\ d_xd_y^2u=d_y^2d_xu,\ d_xd_y^3u=d_y^3d_xu;$$

also, $d_xd_yd_xu=d_x^2d_yu=d_yd_x^2u$; in other words, the successive operations indicated by d_x and d_y may take place in any order.

It will be shown that this is true generally.

230. Continuity of a function of two variables. Let $u=f(x, y)$, and let dx and dy be infinitesimal increments of x

and y. Then $f(x, y)$ is continuous at x, y, if

$$\pounds f(x+dx, y+dy) = f(x, y),$$

when dx and dy approach the limit zero in any manner whatever.

If in Fig. 132 $OA = x$, $AB = y$, AD or $BG = dx$, $BH = dy$, and $BP = f(x, y)$, then $EQ = f(x+dx, y+dy)$. The condition

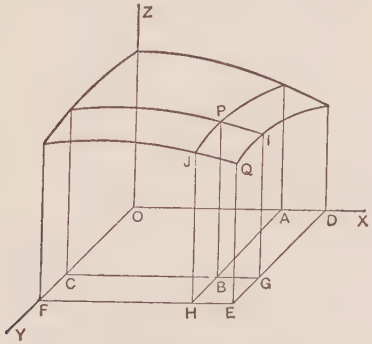

Fig. 132.

of continuity implies that $\pounds EQ = BP$ when E is any point near B in the plane XOY.

In what follows it is assumed that the functions and their derivatives are continuous for the values of the variables under consideration.

231. Let Δ_x indicate an increment produced by the increment dx of x, y being regarded as constant, Δ_y having a corresponding meaning. Then if $u = f(x, y)$, $\Delta_x \Delta_y u = \Delta_y \Delta_x u$.

For, $\quad \Delta_y u = f(x, y+dy) - f(x, y),$

$$\Delta_x \Delta_y u = f(x+dx, y+dy) - f(x+dx, y) - [f(x, y+dy) - f(x, y)]$$

$$= f(x+dx, y+dy) - f(x+dx, y) - f(x, y+dy) + f(x, y). \quad (1)$$

The symmetry of the result shows that it would also be obtained for $\Delta_y\Delta_x u$.

Ex. In Fig. 132 let u be the *volume* $COAB \,.\, P$. Then (1) expresses that

$$HBGE \,.\, Q = FODE \,.\, Q - CODG \,.\, I - FOAH \,.\, J + COAB \,.\, P,$$

which is obvious from the figure, as is also the fact that $HBGE \,.\, Q$ is $\Delta_y\Delta_x u$ as well as $\Delta_x\Delta_y u$.

232. Since u and its derivatives are assumed to be continuous at and near x, y,

$$\Delta_y u = d_y u + I_2$$

(§ 42), where I_2 is an infinitesimal of at least the second order, and

$$\Delta_x\Delta_y u = \Delta_x(d_y u + I_2) = d_x d_y u + I_3,$$

where I_3 is of least the third order. Hence

$$\frac{d_x d_y u}{dx\, dy} = \pounds \frac{\Delta_x\Delta_y u}{dx\, dy}.$$

Similarly, $\dfrac{d_y d_x u}{dx\, dy} = \pounds \dfrac{\Delta_y\Delta_x u}{dx\, dy}$. But $\Delta_y\Delta_x u = \Delta_x\Delta_y u$.

$$\therefore\ d_x d_y u = d_y d_x u.$$

Hence, $d_x d_y{}^2 u = d_x d_y d_y u = d_y d_x d_y u = d_y d_y d_x u = d_y{}^2 d_x u$, and similarly for any combination.

These results may obviously be extended to functions of any number of variables.

The expressions $\dfrac{d_x{}^2 u}{dx^2}, \dfrac{d_y{}^2 u}{dy^2}, \dfrac{d_x d_y u}{dx\, dy}, \dfrac{d_y d_x u}{dy\, dx}, \dfrac{d_x{}^3 d_y u}{dx^3 dy}$, etc. are frequently written

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x\, \partial y}, \frac{\partial^2 u}{\partial y\, \partial x}, \frac{\partial^4 u}{\partial x^3 \partial y}, \text{ etc.}$$

Ex. 1. In Fig. 132, u being as before the volume of OP,

$$\frac{d_x d_y u}{dx\,dy} = \pounds \frac{\Delta_x \Delta_y u}{dx\,dy} = \pounds \frac{HBGE \,.\, Q}{HBGE},$$

i.e., the limit of the mean height of the solid BQ, which limit is BP or z. Hence $d_x d_y u = z\,dx\,dy$, $\therefore$ the volume $= \int\int z\,dx\,dy$ between assigned limits, as in § 179.

2. Verify that $d_y d_x u = d_x d_y u$ or $\dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{\partial^2 u}{\partial x\,\partial y}$ if (1) $u = x \log y$, (2) $u = \sin xy$, (3) $u = \tan^{-1}(y/x)$.

3. If $u = (2x - 3y)^3$, verify that $d_x d_y{}^2 u = d_y{}^2 d_x u$.

4. If $u = r^n \sin n\theta$, $r^2 \dfrac{\partial^2 u}{\partial r^2} + r \dfrac{\partial u}{\partial r} + \dfrac{\partial^2 u}{\partial \theta^2} = 0$.

5. If $u = [(a-x)^2 + (b-y)^2 + (c-z)^2]^{-\frac{1}{2}}$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

6. If $u = f(y + ax) + F(y - ax)$, show that $\dfrac{\partial^2 u}{\partial x^2} = a^2 \dfrac{\partial^2 u}{\partial y^2}$, f and F indicating any continuous functions.

Successive Total Differentials.

233. To find d^2u. We have (§ 45) $du = d_x u + d_y u$, or

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy, \qquad (1)$$

whence $d^2u = d\left(\dfrac{\partial u}{\partial x}\right) dx + \dfrac{\partial u}{\partial x}\,d^2x + d\left(\dfrac{\partial u}{\partial y}\right) dy + \dfrac{\partial u}{\partial y}\,d^2y.$

Find $d\left(\dfrac{\partial u}{\partial x}\right)$ and $d\left(\dfrac{\partial u}{\partial y}\right)$ by substituting $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ for u in (1). The final result is

$$d^2u = \frac{\partial^2 u}{\partial x^2}\,dx^2 + 2\frac{\partial^2 u}{\partial x\,\partial y}\,dx\,dy + \frac{\partial^2 u}{\partial y^2}\,dy^2 + \frac{\partial u}{\partial x}\,d^2x + \frac{\partial u}{\partial y}\,d^2y.$$

d^3u may be found in a similar manner.

Ex. If $u=c$ is a plane curve, $du=0$ and $d^2u=0$. If also $d^2x=0$ (i.e., if x is the independent variable), show that

$$\frac{d^2y}{dx^2}=-\frac{\frac{\partial^2u}{\partial x^2}\left(\frac{\partial u}{y}\right)^2-2\frac{\partial^2u}{\partial x\,\partial y}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial^2u}{\partial y^2}\left(\frac{\partial u}{\partial x}\right)^2}{\left(\frac{\partial u}{\partial y}\right)^3}.$$

Extension of Taylor's Theorem.

234. If in the formula of Taylor's Theorem, § 207, we write $\frac{df(x)}{dx}, \frac{d^2f(x)}{dx^2}, \ldots$ for $f'(x), f''(x), \ldots$, we obtain as an equivalent form

$$f(x+h)=f(x)+\frac{df(x)}{dx}h+\frac{d^2f(x)}{dx^2}\frac{h^2}{2!}+\ldots \qquad (1)$$

Let $f(x, y)$ be a function of x and y, and let x become $x+h$, y for the present remaining unchanged. Then, from (1),

$$f(x+h, y)=f(x, y)+\frac{\partial f(x, y)}{\partial x}h+\frac{\partial^2 f(x, y)}{\partial x^2}\frac{h^2}{2!}+\ldots \qquad (2)$$

If now y becomes $y+k$, (2) becomes

$$f(x+h, y+k)=f(x, y+k)+\frac{\partial f(x, y+k)}{\partial x}h+\frac{\partial^2 f(x, y+k)}{\partial x^2}\frac{h^2}{2!}+\ldots, \qquad (3)$$

and each term may be expanded by Taylor's Theorem as follows, using u for $f(x, y)$:

$$f(x, y+k)=u+\frac{\partial u}{\partial y}k+\frac{\partial^2 u}{\partial y^2}\frac{k^2}{2!}+\ldots,$$

$$\frac{\partial f(x, y+k)}{\partial x}h=\frac{\partial\left[u+\frac{\partial u}{\partial y}k+\ldots\right]}{\partial x}h=\frac{\partial u}{\partial x}h+\frac{\partial^2 u}{\partial x\,\partial y}hk+\ldots,$$

$$\frac{\partial^2 f(x, y+k)}{\partial x^2}\frac{h^2}{2!}=\frac{\partial^2[u+\ldots]}{\partial x^2}\frac{h^2}{2!}=\frac{\partial^2 u}{\partial x^2}\frac{h^2}{2!}+\ldots,$$

whence (3) becomes

$$f(x+h, y+k)=f(x, y)+\left[\frac{\partial u}{\partial x}h+\frac{\partial u}{\partial y}k\right]+\left[\frac{\partial^2 u}{\partial x^2}h^2+2\frac{\partial^2 u}{\partial x\,\partial y}hk+\frac{\partial^2 u}{\partial y^2}k^2\right]\frac{1}{2!}+\ldots \quad (4)$$

If $D\equiv\frac{\partial}{\partial x}h+\frac{\partial}{\partial y}k$, the *form* of (4) is the same as

$$f(x+h, y+k)=u+Du+\frac{D^2u}{2!}+\frac{D^3u}{3!}+\ldots$$
$$=\left(1+D+\frac{D^2}{2!}+\frac{D^3}{3!}+\ldots\right)u=e^{D}u.$$

A similar result would apply to functions of three or more variables.

Ex. 1. **Euler's theorem on homogeneous functions.** *Def.* A function u or $f(x, y)$ is said to be homogeneous and of the degree n when $f(mx, my)=m^n f(x, y)$, where m is any number. For example: $2x^3+y^3$, x^2-xy+y^2, $(x^2+y^2)/(x^2-y^2)$, $(x-y)/(x^2+y^2)$, $ax^{\frac{1}{2}}+by^{\frac{1}{2}}$.

Let $m=1+r$. Then

$$f(x+rx, y+ry)=(1+r)^n f(x, y).$$

Expanding the first member by Taylor's Theorem and the second by the Binomial Theorem,

$$u+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)r+\left(x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 u}{\partial y^2}\right)\frac{r^2}{2!}+\ldots$$
$$=\left[1+nr+n(n-1)\frac{r^2}{2!}+\ldots\right]u.$$

Equating like powers of r,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu,$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

The results may evidently be extended to higher derivatives, and to functions of three or more variables.

2. If u is homogeneous, show that

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\,\partial y}=(n-1)\frac{\partial u}{\partial x},$$

$$x\frac{\partial^2 u}{\partial x\,\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}.$$

Maxima and Minima from Taylor's Theorem.

235. By the aid of Taylor's Theorem we may verify and extend the conclusions of Chapter XVII for maxima and minima.

If a is a value of x for which any function $f(x)$ is a max. or a min., and h any small quantity, it is plain that $f(a+h)-f(a)$ and $f(a-h)-f(a)$ must have the same sign, viz., + for a min. and − for a max. Now

$$f(a+h)-f(a)=f'(a)h+f''(a)\frac{h^2}{2!}+f'''(a)\frac{h^3}{3!}+\ldots$$

and $$f(a-h)-f(a)=-f'(a)h+f''(a)\frac{h^2}{2!}-f'''(a)\frac{h^3}{3!}+\ldots;$$

and by taking h small enough the sign of the right-hand side will depend upon that of the first term which does not vanish. Hence there cannot be a max. or a min. unless $f'(a)=0$, and there will then be a max. if $f''(a)$ is − and a min. if $f''(a)$ is +. But if $f''(a)$ also $=0$, there cannot be a max. or a min. unless $f'''(a)$ also $=0$, and there will be a max. or a min. according as $f^{(iv)}(a)$ is − or +. It will thus be seen that there cannot be a max. or a min. unless the first derivative which does not vanish is of an even order, and that $f(a)$ will be a max. or a min. according as this derivative is − or +.

For a similar reason, from § 234 (4), a function u or $f(x, y)$ of two independent variables is a max. or a min

for values a and b of the variables if a and b satisfy $\partial u/\partial x=0$ and $\partial u/\partial y=0$, and at the same time

$$\frac{\partial^2 u}{\partial x^2}h^2+2\frac{\partial^2 u}{\partial x\,\partial y}hk+\frac{\partial^2 u}{\partial y^2}k^2 \qquad (1)$$

is not zero, and is *in sign* independent of the values of h and k. These conditions are satisfied if

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2 \text{ is } +.$$

For, $(1)\equiv Ah^2+2Bhk+Ck^2\equiv\dfrac{(Ah+Bk)^2+(AC-B^2)k^2}{A}$,

and $\therefore$ has the same sign as A if $AC-B^2$ is $+$.

Hence if $\dfrac{\partial u}{\partial x}=0$, $\dfrac{\partial u}{\partial y}=0$, and $\dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial^2 u}{\partial y^2}-\left(\dfrac{\partial^2 u}{\partial x\,\partial y}\right)^2$ is $+$, u is a max. or a min. according as $\dfrac{\partial^2 u}{\partial x^2}$ is $-$ or $+$.

Similarly for a function of three independent variables we must have $\partial u/\partial x=0$, $\partial u/\partial y=0$, $\partial u/\partial z=0$, to solve for x, y, and z.

Ex 1. $u=x^2+xy+y^2+x-2y+4$.

$$\partial u/\partial x=2x+y+1, \quad \partial u/\partial y=x+2y-2.$$

Putting these $=0$ and solving for x and y we get $x=-\frac{4}{3}$, $y=\frac{5}{3}$, which make u a min., viz., $1\frac{2}{3}$.

2. The max. value of $(2ax-x^2)(2by-y^2)$ is a^2b^2.

3. The max. value of $(x-1)(y-1)(x+y-1)$ is $\frac{1}{27}$.

4. The min. value of x^3+y^3-3axy is $-a^3$.

5. The max. or min. value of $ax^2+2hxy+by^2+2gx+2fy+c$ is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \div \begin{vmatrix} a & h \\ h & b \end{vmatrix}.$$

6. Find a point such that the sum of the squares of its distances from any number of given points (a_1, b_1), (a_2, b_2), ... may be a min. *Ans.* $\left(\dfrac{1}{n}\Sigma a, \dfrac{1}{n}\Sigma b\right)$, the centre of mean position.

7. Given $r_1 = a_1x + b_1y + c_1$, $r_2 = a_2x + b_2y + c_2$, . . . , show that the values of x and y which make $r_1^2 + r_2^2 + r_3^2 + \ldots$ a min. are obtained by solving the equations

$$x\Sigma(a^2) + y\Sigma(ab) + \Sigma(ac) = 0,$$
$$x\Sigma(ab) + y\Sigma(b^2) + \Sigma(bc) = 0.$$

These are the *normal equations* in the method of Least Squares.

8. To make with the smallest possible amount of sheet metal an open rectangular box of given volume, show that the length and breadth must each be double of the depth.

9. To cut circular sectors from the angles of a triangle so as to leave the greatest area with a given perimeter, show that the radii must be equal.

CHAPTER XLIV.

DIFFERENTIAL EQUATIONS* OF THE FIRST ORDER.

236. A differential equation is an equation containing one or more derivatives. The derivatives are usually represented by the corresponding differentials.

The *order* of a differential equation is the order of the highest derivative in the equation. The *degree* of the equation is the degree of the highest derivative when the equation is free from fractions and radicals affecting the derivatives.

Ex. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$ is of the second order and first degree.

$\left(\frac{dy}{dx}\right)^2+2\frac{dy}{dx}+y=0$ is of the first order and second degree.

Partial differential equations are those which contain partial derivatives; other differential equations are called *ordinary*.

237. Ordinary differential equations frequently appear in the statement of problems in Geometry, Mechanics, Physics, etc., but for our present purpose they may be supposed to arise from the elimination of constants.

Ex. 1. $y=mx$, $\frac{dy}{dx}=m=\frac{y}{x}$. $\therefore \frac{dy}{dx}=\frac{y}{x}$.

This is a differential equation of the first order obtained by differentiating, and eliminating the constant m. It may be

* For further information relating to differential equations see Murray's *Differential Equations* (Longmans), from which some of the examples of this and the following chapter have been taken.

called the differential equation of all straight lines passing through the origin.

2. $y=mx+b$, $\frac{dy}{dx}=m$, $\frac{d^2y}{dx^2}=0$.

$\therefore$ $d^2y/dx^2=0$ is the differential equation of all straight lines. Two constants, m and b, have been eliminated. The equation is of the second order.

3. $y=ax^2+b$, $\frac{dy}{dx}=2ax$, $\frac{d^2y}{dx^2}=2a=\frac{1}{x}\cdot\frac{dy}{dx}$.

$\therefore$ $\frac{d^2y}{dx^2}=\frac{1}{x}\frac{dy}{dx}$, an equation of the second order.

The elimination of n constants requires $n+1$ equations viz., the original equation and n derived equations. Hence the order of the resulting differential equation is equal to the number of constants eliminated.

Eliminate a, b, c from the following equations:

	Ans.
4. $y=ae^{mx}+be^{-mx}$.	$d^2y/dx^2=m^2y$.
5. $y=a\sin mx+b\cos mx$.	$d^2y/dx^2=-m^2y$.
6. $y=ax^2+bx+c$.	$d^3y/dx^3=0$.
7. $2y=cx^2+\frac{1}{c}$.	$x\left(\frac{dy}{dx}\right)^2-2y\frac{dy}{dx}+x=0$.
8. $y^2=(x-c)^3$.	$8(dy/dx)^3=27y$.
9. $y=ax^2+bx$.	$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$.

238. An *integral* or *solution* of a differential equation is a relation between the variables which satisfies the equation.

Ex. $y=A\cos x$, $y=B\sin x$, $y=A\sin x+B\cos x$, $y=a\sin(x+b)$, $y=a\cos(x+b)$, are all solutions of the equation $\frac{d^2y}{dx^2}+y=0$.

The solution which contains a number of arbitrary constants equal to the order of the given equation is said to be a *complete integral* or *general solution*. Particular solutions are those which may be obtained from the general solution by assigning values to the constants.

Separation of the Variables.

239. In some cases no special method of solution is required. An algebraical rearrangement of the terms will cause the equation to take the form

$$f_1(x)dx+f_2(y)dy=0,$$

and each term may be integrated.

Ex. 1. $2x^2y\,dy=(1+x^2)dx$ is the same as

$$2y\,dy=\left(\frac{1+x^2}{x^2}\right)dx=\frac{dx}{x^2}+dx.$$

Hence, integrating, $y^2=-x^{-1}+x+c$,
where c may have any assigned value (see § 96).

2. $y\,dx-x\,dy=dx+x^2dy$, or $\dfrac{dy}{y-1}=\dfrac{dx}{x^2+x}$.

Integrating, $\log(y-1)=\log x-\log(x+1)+\log c$,*
$\therefore\ y-1=cx/(x+1)$.

3. $(x^2+y^2-y)dx+x\,dy=0$ is the same as

$$\left(1+\frac{y^2}{x^2}\right)dx+\frac{x\,dy-y\,dx}{x^2}=0, \quad \text{or} \quad dx+\frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}=0.$$

Hence $x+\tan^{-1}(y/x)=c$, or $y=x\tan(c-x)$.

4. $x^2y\,dy+m\,dx=0$. *Ans.* $y^2=2m/x+c$.
5. $(x-y^2x)dx+(y-x^2y)dy=0$. $x^2+y^2=x^2y^2+c$.
6. $x\,dy=(x^3+y)dx$. $y=\frac{1}{2}x^3+cx$.
7. $(x^2y+x)dy+(xy^2-y)dx=0$. $xy+\log(y/x)=c$.
8. $y\,dy=(\sqrt{x^2+y^2}-x)dx$. $y^2=2cx+c^2$.

240. The separation of the variables is sometimes assisted by a substitution. The following is an important case.

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* In order to simplify the final result the constant may be written in the form $\log c$, or in any other form which permits of any arbitrary value.

Homogeneous equations. If the given equation is of the form

$$f_1(x, y)dx + f_2(x, y)dy = 0,$$

where the functions are homogeneous in x and y, and of the same degree, let $y = vx$. In the new equation in terms of v and x the variables will be separable. In some cases the substitution $x = vy$ may be simpler.

Ex. 1. $xy^2dy = (x^3 + y^3)dx$. If $y = vx$,

$$v^2(x\,dv + v\,dx) = (1 + v^3)dx, \quad \text{or} \quad v^2dv = dx/x.$$

$$\therefore \tfrac{1}{3}v^3 = \log cx, \quad \text{or} \quad y^3 = 3x^3 \log cx.$$

2. $(x^2 - 2y^2)dx + 2xy\,dy = 0$. *Ans.* $y^2 = -x^2 \log cx$.
3. $(x^2 + y^2)dx = 2xy\,dy$. $y^2 = x^2 + cx$.
4. $y^2dx + x^2dy = xy\,dy$. $x = y/\log cy$.
5. $(x + y)dy + (x - y)dx = 0$.

Ans. $\tan^{-1}(y/x) + \log\sqrt{x^2 + y^2} = c$.

6. Show that the homogeneous equation

$$f_1(x, y)dx + f_2(x, y)dy = 0,$$

or

$$f_1(1, v)dx + f_2(1, v)dy = 0$$

becomes

$$\frac{dx}{x} + \frac{f_2(1, v)dv}{f_1(1, v) + vf_2(1, v)} = 0.$$

7. Show that an equation of the form

$$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$$

can be integrated by the substitution $y = v/x$.

241. An equation of the form

$$(ax + by + c)dx + (a'x + b'y + c')dy = 0 \qquad (1)$$

is not homogeneous, but may be reduced to a homogeneous equation by the method of the following example.

Ex. 1. $(3x - y - 5)dx + (x + y + 1)dy = 0$.
Let $x = X + h$, $y = Y + k$. Then, substituting,

$$(3X - Y + 3h - k - 5)dX + (X + Y + h + k + 1)dY = 0.$$

Take h and k so that $3h-k-5=0$ and $h+k+1=0$. $\therefore\ h=1$, $k=-2$. The equation is now

$$(3X-Y)dX+(X+Y)dY=0,$$

which is homogeneous. Let $Y=vX$. Then

$$\frac{1+v}{3+v^2}dv+\frac{dX}{X}=0,$$

whence, $\quad \frac{1}{\sqrt{3}}\tan^{-1}\frac{v}{\sqrt{3}}+\frac{1}{2}\log(3+v^2)+\log X=c.$

But $v=\frac{Y}{X}=\frac{y-k}{x-h}=\frac{y+2}{x-1}$.

$$\therefore\ \frac{1}{\sqrt{3}}\tan^{-1}\frac{y+2}{\sqrt{3}\,(x-1)}+\frac{1}{2}\log[3(x-1)^2+(y+2)^2]=c.$$

Hence to solve an equation of the form (1), drop the c and c', solve the resulting homogeneous equation, and in the result substitute $x-h$ for x and $y-k$ for y, where h and k are the roots of the simultaneous equations $ax+by+c=0$, $a'x+b'y+c'=0$.

The method would fail if $a'x+b'y=k(ax+by)$, k being any constant; but the equation could then be solved by the substitution $v=ax+by$ and the elimination of y or x.

Ex. 2. $(3x-2y-5)dx+(2x-3y-5)dy=0$.

Ans. $(x+y)^5\,(x-y-2)=c$.

3. $(2x-y)dx-(4x-2y-1)dy=0$.

Ans. $3(x-2y)+\log(3y-6x+2)=c$.

Exact Differential Equations.

242. The result of differentiating $f(x,y)=c$ or $u=c$ is (§ 45)

$$M\,dx+N\,dy=0,$$

where $\quad M=\partial u/\partial x$ and $N=\partial u/\partial y$.

Hence $\quad \frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}$ and $\frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}$.

But $\frac{\partial^2 u}{\partial y\,\partial x}=\frac{\partial^2 u}{\partial x\,\partial y}$ (§ 232). $\quad\therefore\ \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Conversely, if $M\,dx+N\,dy=0$ is an equation such that $\partial M/\partial y=\partial N/\partial x$ the equation is an exact differential equation, i.e., one obtained directly by differentiating without further change.

The x-integral of $M\,dx$ contains all the terms of u except those which are independent of x. Hence to integrate an exact equation, integrate $M\,dx$ with regard to x, integrate with regard to y those terms of $N\,dy$ which contain y only, and put the sum of the results equal to a constant.

Ex. 1. $(4x^3+6x^2y)dx+(2x^3-2y)dy=0$.
Here $\partial M/\partial y=6x^2$, and $\partial N/\partial x=6x^2$.
Hence the solution is $x^4+2x^3y-y^2=c$.

2. $(2-2xy-y^2)dx-(x+y)^2dy=0$. *Ans.* $2x-x^2y-xy^2-\frac{1}{3}y^3=c$.

3. $(x^3+y)dx+x\,dy=0$. $\frac{1}{4}x^4+xy=c$.

243. Integrating factor. After forming a differential equation the result can sometimes be simplified by dividing by a variable factor. Conversely, a differential equation may sometimes be made exact by multiplying by a factor.

Ex. 1. $(1+xy)y\,dx+(1-xy)x\,dy=0$ is not exact, since $\partial M/\partial y=1+2xy$ and $\partial N/\partial x=1-2xy$. Multiplying by $1/x^2y^2$ the equation becomes

$$\left(\frac{1}{x^2y}+\frac{1}{x}\right)dx+\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0.$$

Here $\partial M/\partial y=-1/x^2y^2$ and $\partial N/\partial x=-1/x^2y^2$, hence the equation is exact. The solution is therefore

$$-\frac{1}{xy}+\log x-\log y=c.$$

2. $(x^2+y^2+1)dx-2xy\,dy=0$, factor $\frac{1}{x^2}$. *Ans.* $x-\frac{y^2}{x}-\frac{1}{x}=c$.

3. $2(x\,dy+y\,dx)=xy\,dx$, factor $\frac{1}{xy}$. $y=\frac{1}{x}e^{\frac{x+c}{2}}$.

244. Linear equations of the first order. A differential equation is linear when it contains the first power only of the function and its derivatives. The linear equation of the first order is of the form

$$\frac{dy}{dx}+Py=Q, \quad \text{or} \quad dy+Py\,dx=Q\,dx,$$

where P and Q are independent of y. Of this equation $e^{\int P\,dx}$ is an integrating factor.

For $e^{\int P\,dx}\,.\,dy+y\,.\,e^{\int P\,dx}\,.\,P\,dx=e^{\int P\,dx}Q\,dx$

is the same as $\quad d\left(ye^{\int P\,dx}\right)=e^{\int P\,dx}Q\,dx.$

$$\therefore\ ye^{\int P\,dx}=\int e^{\int P\,dx}Q\,dx+c.$$

Hence the solution of $\frac{dy}{dx}+Py=Q$ is

$$y=e^{-\int P\,dx}\left(\int e^{\int P\,dx}Q\,dx+c\right).$$

Ex. 1. $\dfrac{dy}{dx}-\dfrac{n}{x}y=e^x x^n.$

$$\int P\,dx=-n\log x=\log x^{-n}. \quad \therefore\ e^{\int P\,dx}=x^{-n}.$$

$$\therefore\ y=x^n\left(\int e^x dx+c\right)=x^n(e^x+c).$$

2. $\dfrac{dy}{dx}+\dfrac{ny}{x}=\dfrac{a}{x^n}.$ *Ans.* $y=\dfrac{ax+c}{x^n}.$

3. $dy/dx+y=e^{-x}.$ $y=e^{-x}(x+c).$

4. $x\,dy/dx=2y+x+1.$ $y=-x-\frac{1}{2}+cx^2.$

5. $dy/dx+y\cos x=\sin x\cos x.$ $y=\sin x-1+ce^{-\sin x}.$

6. $2xy\dfrac{dy}{dx}=y^2-x,$ or $\dfrac{d(y^2)}{dx}-\dfrac{y^2}{x}=-1.$

Ans. $y=\sqrt{cx-x\log x}.$

7. $\dfrac{dy}{dx}+\dfrac{1-2x}{x^2}y=1.$ $y=x^2(1+ce^{x^{-1}}).$

8. $(1+x^2)dy=(a+xy)dx$. *Ans.* $y=ax+c\sqrt{1+x^2}$.

9. $\dfrac{dy}{dx}+ay=b\sin mx$.

Ans. $y=\dfrac{b}{a^2+m^2}(a\sin mx-m\cos mx)+ce^{-ax}$.

245. Bernoulli's equation. An equation of the form

$$\frac{dy}{dx}+Py=Qy^n,$$

where P and Q are independent of y, may be made linear by the substitution $\dfrac{1}{y^{n-1}}=z$. It is best to divide through by y^n before substituting.

Ex. 1. $x\dfrac{dy}{dx}+y=y^2\log x$, or $\dfrac{1}{y^2}\dfrac{dy}{dx}+\dfrac{1}{x}\cdot\dfrac{1}{y}=\dfrac{\log x}{x}$.

Let $\dfrac{1}{y}=z$, then $-\dfrac{1}{y^2}dy=dz$.

$$\therefore \frac{dz}{dx}-\frac{z}{x}=-\frac{\log x}{x},$$

which is linear. Solving,

$$z=\log x+1+cx. \quad \therefore y=\frac{1}{\log x+1+cx}.$$

2. $(1-x^2)\dfrac{dy}{dx}-xy=3xy^2$. *Ans.* $y=\dfrac{1}{c\sqrt{1-x^2}-3}$.

3. $\dfrac{dy}{dx}+xy=x^3y^3$. $y=\dfrac{1}{\sqrt{1+x^2+ce^{x^2}}}$.

Equations of the First Order but not of the First Degree.

246. Let dy/dx be called p.

If possible solve the equation for p.

Ex. 1. $p^2-(x-3y)p-3xy=0$, or $(p-x)(p+3y)=0$.

The equation is satisfied if

$$p-x=0, \text{ or } p+3y=0;$$

i.e., if

$$\frac{dy}{dx}-x=0, \text{ or } \frac{dy}{dx}+3y=0;$$

or, integrating, if

$$y-\tfrac{1}{2}x^2+c=0, \quad \text{or} \quad y+ce^{-3x}=0.$$

These equations may be regarded as the solutions of the given equation, or they may be combined into

$$(y-\tfrac{1}{2}x^2+c)(y+ce^{-3x})=0.$$

2. $p^2-9p+18=0$. *Ans.* $(y-6x+c)(y-3x+c)=0$.

3. $p^3=ax^4$. $343(y+c)^3=27ax^7$.

247. When it is not possible or convenient to solve for p we may be able to solve for y, then, differentiating throughout and substituting $p\,dx$ for dy, obtain a new equation in p and x which we may be able to integrate and thus find the relation connecting p and x. From this result and the given equation we may be able to eliminate p and thus obtain the relation connecting x and y, or, if this elimination is not convenient or possible, x and y may be left in terms of p as a third variable.

Ex. 1. $p^2x-2py+x=0$, or $2y=px+x/p$.

Differentiating, substituting $p\,dx$ for dy, and reducing,

$$dp/p=dx/x, \quad \therefore\ p=cx.$$

Hence, substituting in the given equation, $2y=cx^2+\dfrac{1}{c}$.

2. $p^2-py+1=0$. *Ans.* $x=\dfrac{1}{2p^2}+\log p+c, \quad y=p+\dfrac{1}{p}$.

248. Instead of solving for y we may be able to solve for x, then differentiate throughout and substitute dy/p for dx, and proceed as above.

Ex. 1. $p^2-px+1=0$. *Ans.* $x=p+\dfrac{1}{p}, \quad y=\tfrac{1}{2}p^2-\log p+c$.

2. $p^2y+2px=y$. $y^2=2cx+c^2$.

3. $p^3-p^2x+1=0$. $x=p+\dfrac{1}{p^2}, \quad y=\dfrac{p^2}{2}+\dfrac{2}{p}+c$.

249. Clairaut's equation. Singular solution. The method of § 247 is applicable to Clairaut's equation,

$$y = px + f(p). \tag{1}$$

Differentiating and substituting $p\,dx$ for dy,

$$dp[x + f'(p)] = 0.$$

From $dp = 0$ we have $p = c$, and substituting in the given equation,

$$y = cx + f(c), \tag{2}$$

the general solution.

The equation is also satisfied if $x + f'(p) = 0$, and eliminating p from this and the given equation we have another solution which is not contained in the general solution and which does not contain any arbitrary constant. Such a solution is called a singular solution.

The general solution (2) represents, for various values of c, a family of straight lines. The singular solution represents the envelope of these straight lines. For the envelope of the family of lines is obtained (§ 157) by eliminating c from

$$y = cx + f(c) \quad \text{and} \quad 0 = x + f'(c),$$

the same equations (with c instead of p) as those from which the singular solution is obtained.

Ex. 1. $y = px + a/p$. The general solution is $y = cx + a/c$. Also $x + f'(p) = 0$ is $x - a/p^2 = 0$. $\therefore\ p^2 = a/x$. Substituting in the given equation we obtain $y^2 = 4ax$, the singular solution.

2. Find the singular solution of $y = px + p^2$. *Ans.* $x^2 + 4y = 0$.

3. Find the general and singular solutions of

$$y = px + a\sqrt{1 + p^2}.$$

Ans. $y = cx + a\sqrt{1 + c^2}$, $x^2 + y^2 = a^2$.

4. Solve $y = -xp + x^4p^2$. Let $x = z^{-1}$. *Ans.* $y = c/x + c^2$.

Examples.

1. $dy/dx + y \cot x = 2 \cos x$. *Ans.* $y = \sin x + c \operatorname{cosec} x$.
2. $x^2 dy + x^2y^2 dx + 4 dy = 0$. $y^{-1} = x - 2 \tan^{-1}\frac{1}{2}x + c$.
3. $p^2 = px - y$. $y = cx - c^2$.
4. $px^2 + y^2 = xy$. $x = y(c + \log x)$.
5. $(2x^2 + 4xy)dx + (2x^2 - y^2)dy = 0$. $2x^3 + 6x^2y - y^3 = c$.
6. $px + y = x^3y^6$. $y^{-5} = \frac{5}{2}x^3 + cx^5$.
7. $(x^2 - y^2 + 2x)dx = 2y\, dy$. $x^2 - y^2 = ce^{-x}$.
8. $(2ax + hy + f)dx + (hx + 2by + g)dy = 0$.
   $ax^2 + by^2 + hxy + fx + gy = c$.
9. $2xy\, dx + (y^2 - 3x^2)dy = 0$. $x^2 - y^2 = cy^3$.
10. $x\, dx\, dy = y\, dx^2 + 2\, dy^2$. $cx = c^2y + 2$.
11. $x^2p^2 = 2xyp + 3y^2$. $(xy - c)(y - cx^3) = 0$.
12. $y^3dx = (2x^2 + 3xy^2)dy$. $2xy + y^3 = cx$.
13. $(y - a)dx = (x^2 + x)dy$. $(x + 1)y = a + cx$.
14. $x^2(y - px) = p^2y$. Let $y^2 = v$, $x^2 = z$. $y^2 = cx^2 + c^2$.
15. Find the curve in which the subnormal is constant and $= a$.

The condition is that $y\, dy/dx = a$.

Ans. The parabola $y^2 = 2ax + c$.

16. Find the curve in which the subtangent is constant and $= a$.

Ans. $y = ce^{x/a}$.

17. Find the curve in which the perpendicular on the tangent from the foot of the ordinate is constant and $= a$.

Ans. The catenary $y = a \cosh (x + c)/a$.

18. Find the curve in which the area bounded by the curve, two ordinates, and the x-axis is proportional to the length of the bounding arc.

$[y\, dx \propto dA = d(as) = a\sqrt{dx^2 + dy^2}]$.

Ans. The catenary $y = a \cosh (x + c)/a$.

19. Find the curve in which $\log s = x$.

Ans. $y = \sqrt{e^{2x} - 1} - \sec^{-1}e^x + c$.

20. Find the curve in which $\psi = \frac{1}{2}\theta$ (§ 136).

Ans. The cardioid $r = c(1 - \cos \theta)$.

21. To find the amount, at compound interest due and added to the principal *all the time*, of a sum of money P, in t years, at rate r per unit per annum.

Let x be the amount at time t. When the interest is due at the

end of an interval Δt, $xr\Delta t$ = interest = Δx. Hence in the present case

$$xr = \underset{\Delta t \doteq 0}{\mathrm{L}}(\Delta x/\Delta t) = dx/dt.$$

Integrating, $\log x = rt + c$. But $x = P$ when $t = 0$. $\therefore \log P = c$. $\therefore \log x = rt + \log P$, or $x = Pe^{rt}$.

For example, for 5 years at 4 per cent per annum, $e^{rt} = e^{\cdot 04 \times 5} = 1{\cdot}2214$. The result may be compared with $1 + rt = 1{\cdot}2000$, the amount at simple interest, and $(1 + r)^t = 1{\cdot}2167$, the amount at compound interest due yearly.

22. **Orthogonal trajectories.** By giving different values to a in the equation $x^2 + y^2 = 2ay$ we have a family of circles touching the x-axis at the origin. Let it be required to find another family of curves all of which cut all of the given curves at right angles. Such curves are called orthogonal trajectories of the given curves.

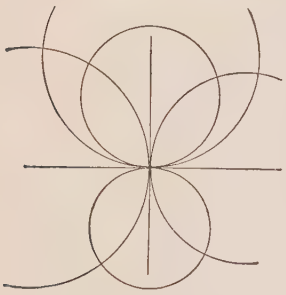

Fig. 133.

Differentiating $x^2 + y^2 = 2ay$ and eliminating a we have

$$2xy + (y^2 - x^2)\frac{dy}{dx} = 0,$$

the differential equation of the given curves. At a point (x, y) where one of the required curves intersects one of the given curves, dy/dx of the given curve $= -dx/dy$ of the new curve, since they intersect at right angles. Hence the differential equation of the required curves is

$$2xy - (y^2 - x^2)\frac{dx}{dy} = 0,$$

a homogeneous equation of which the solution is

$$x^2 + y^2 = cx.$$

The required curves are therefore circles touching the y-axis at the origin.

23. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$. *Ans.* The ellipses $2x^2 + y^2 = c^2$.

24. Find the orthogonal trajectories of

(1) The rectangular hyperbolas $x^2 - y^2 = a^2$. *Ans.* $xy = c^2$.

(2) The straight lines $y = mx$.

(3) The curves $y = ax^n$. *Ans.* The conics $x^2 + ny^2 = c^2$.

25. Show that the differential equation of the orthogonal trajectories of a family of curves represented by a polar equation is found by substituting $-r^2 d\theta/dr$ for $dr/d\theta$ in the differential equation of the given curves.

26. Find the orthogonal trajectories of $r^m \cos m\theta = a^m$.

Ans. $r^m \sin m\theta = c^m$.

27. Find the curves which make an angle 45° with the parabolas $r(1+\cos\theta)=2a$. *Ans.* The parabolas $r(1+\sin\theta)=2c$.

28. If E = the extraneous electromotive force in a circuit having resistance R and inductance L,

$$L\frac{di}{dt}+Ri=E, \tag{1}$$

where i is the current at time t. The equation is linear, and if E is constant the solution is

$$i=\frac{E}{R}+ce^{-\frac{R}{L}t}.$$

If $i=0$ when $t=0$, $\quad c=-E/R.$

$$\therefore\ i=\frac{E}{R}-\frac{E}{R}e^{-\frac{R}{L}t}.$$

The second term soon becomes very small; the current is then practically constant and $=E/R$, as if there had been no induction.

For an alternating E.M.F. let $E=E_m \sin nt$, where E_m is the maximum value, and the time of a period is $2\pi/n$. The solution of (1) is now

$$i=\frac{E_m}{R^2+L^2n^2}(R\sin nt-Ln\cos nt)+ce^{-\frac{R}{L}t}$$

$$=\frac{E_m}{\sqrt{R^2+L^2n^2}}\sin(nt-\theta)+ce^{-\frac{R}{L}t},$$

where $\theta=\tan^{-1}(Ln/R)$.

In a short time the exponential term becomes very small and the current is represented by a harmonic function of the same period as the E.M.F.

θ is the *lag* of the current behind the voltage. The *impedance* (amplitude of voltage ÷ amplitude of current) is $\sqrt{R^2+L^2n^2}$.

CHAPTER XLV.

DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

250. Equations of the second order must contain d^2y/dx^2 and may contain dy/dx, x, and y.

We shall first consider equations which do not contain y, and then equations which do not contain x. It will be seen that such equations may be reduced to equations of the first order.

251. Equations which do not contain y directly. Let $p=dy/dx$. The equation will be reduced to one of the first order in p and x.

Ex. 1. $\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}=x$, or $\dfrac{dp}{dx}-p=x$.

The latter is a linear equation of the first order. Hence, § 244,

$$p=e^x\left(\int e^{-x}x\,dx+c\right)=-(1+x)+ce^x$$

$$\therefore\ y=\int p\,dx=-\tfrac{1}{2}(1+x)^2+ce^x+c_1.$$

2. $\dfrac{d^2y}{dx^2}=a\left(\dfrac{dy}{dx}\right)^2$. *Ans.* $y=c_1-\dfrac{1}{a}\log(ax+c)$.

3. $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}=0$. $y=c\log x+c_1$.

4. $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}=0$. $y=\dfrac{c}{x}+c_1$.

5. $\dfrac{d^2y}{dx^2}-\left(\dfrac{dy}{dx}\right)^2=1$. $y=c_1+\log\sec(x+c)$.

The same method may be applied to any equation in which y appears only in two derivatives whose orders differ by unity.

Ex. $\frac{d^3y}{dx^3}\frac{d^2y}{dx^2}=2$. First let $\frac{d^2y}{dx^2}=q$. *Ans.* $y=\frac{8}{15}(x+c)^{\frac{5}{2}}+c_1x+c_2$.

If dy/dx is absent as well as y we may integrate directly.

Ex. $\frac{d^2y}{dx^2}=x^3$, $\frac{dy}{dx}=\frac{x^4}{4}+c$, $y=\frac{x^5}{20}+cx+c_1$.

The same method will apply to any equation of the form

$$d^ny/dx^n=f(x).$$

Ex. 1. $d^3y/dx^3=\sin x$. *Ans.* $y=\cos x+cx^2+c_1x+c_2$.

2. $d^ny/dx^n=(x+n)e^x$.

Ans. $y=xe^x+c_1x^{n-1}+c_2x^{n-2}+\ldots+c_n$.

252. Equations which do not contain x directly. Let $\frac{dy}{dx}=p$, then $\frac{d^2y}{dx^2}=\frac{dp}{dy}\frac{dy}{dx}=p\frac{dp}{dy}$. The resulting equation is of the first order in p and y.

Ex. 1. $\frac{d^2y}{dx^2}=-a^2y$, or * $p\frac{dp}{dy}=-a^2y$.

$\therefore$ $\frac{1}{2}p^2=-\frac{1}{2}a^2y^2+\text{const.}$, or say $p^2=a^2(c^2-y^2)$.

$\therefore$ p or $\frac{dy}{dx}=a\sqrt{c^2-y^2}$, or $\frac{dy}{\sqrt{c^2-y^2}}=a\,dx$.

$\therefore$ $\sin^{-1}y/c=ax+c_1$, or $y=c\sin(ax+c_1)$.

The result may also be written

$$y=A\sin ax+B\cos ax,$$

where A and B are arbitrary constants.

2. $d^2y/dx^2=a^2y$.

The result may be in any of the following forms:

$y=ce^{ax}+c_1e^{-ax}$, $y=c\sinh ax+c_1\cosh ax$,

$y=c\sinh(ax+c_1)$, $y=c\cosh(ax+c_1)$.

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* Or without using p, multiply by $2dy$; then $\frac{2dy\,d^2y}{dx^2}=-a^2\,2y\,dy$,

$\therefore$ $\left(\frac{dy}{dx}\right)^2=-a^2y^2+\text{const.}$, etc.

3. $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0.$ *Ans.* $y^2=cx+c_1.$

4. $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1.$ $y^2=x^2+cx+c_1.$

The Operator D.

253. Let Dy represent dy/dx, the symbol D indicating the operation of taking the derivative of y. If a is a constant, $d(ay)/dx=a\,dy/dx$, hence $D(ay)=a\,Dy$.

Let $D\,.\,Dy$, i.e., d^2y/dx^2, be represented by D^2y. Also let $dy/dx-ay$ or $Dy-ay$ be expressed by $(D-a)y$. Then $D-a$ indicates the operation of taking the derivative of a function and subtracting a times that function. Hence, a and b being constants,

$$\begin{aligned}(D-b)\,.\,(D-a)y&=D(D-a)y-b(D-a)y\\&=D^2y-a\,Dy-b\,Dy+aby.\end{aligned}$$

Let this be written

$$[D^2-(a+b)D+ab]y.$$

In the same way it may be shown that

$$(D-a)\,.\,(D-b)y=[D^2-(a+b)D+ab]y,$$

and hence the order of the operations indicated by $D-a$ and $D-b$ does not affect the result, and the successive operations may be replaced by an operation which is indicated by

$$D^2-(a+b)D+ab.$$

254. Suppose $Dy=X$, where X is a constant or a function of x, and let the inverse operation by which y is obtained from X be denoted by D^{-1}. Then $y=D^{-1}X$. But $dy/dx=X$.

$$\therefore\ y=\int X\,dx+c.$$

$$\therefore\ D^{-1}X=\int X\,dx+c.$$

Again, suppose $(D-a)y=X$, and denote the operation inverse to $D-a$ by $(D-a)^{-1}$. Then $y=(D-a)^{-1}X$. But $(D-a)y=X$ is the same as $dy/dx-ay=X$, and the solution of this linear equation is (§ 244)

$$y=e^{ax}\left(\int e^{-ax}X\,dx+c\right).$$

$$\therefore\ (D-a)^{-1}X=e^{ax}\left(\int e^{-ax}X\,dx+c\right).$$

If $X=0$, $(D-a)^{-1}0=ce^{ax}$.

Ex. 1. $D^{-1}0=c$, $D^{-2}0\equiv D^{-1}\,.\,D^{-1}0=cx+c_1$.

2. $D^{-1}a=ax+c$, $D^{-2}a=\frac{1}{2}ax^2+cx+c_1$.

3. $(D-a)^{-1}b=-b/a+ce^{ax}$.

4. $(D-a)^{-1}x=-\dfrac{1}{a}\left(x+\dfrac{1}{a}\right)+ce^{ax}$.

5. $(D-a)^{-1}x^n$, n a positive integer,

$$=-\frac{1}{a}\left(x^n+\frac{nx^{n-1}}{a}+\frac{n(n-1)x^{n-2}}{a^2}+\ldots+\frac{n!}{a^n}\right)+ce^{ax}.$$

6. $(D-a)^{-1}\sin mx=-\dfrac{a\sin mx+m\cos mx}{a^2+m^2}+ce^{ax}$.

7. $(D-a)^{-1}\cos mx=\dfrac{-a\cos mx+m\sin mx}{a^2+m^2}+ce^{ax}$.

8. $(D-a)^{-1}e^{nx}=\dfrac{e^{nx}}{n-a}+ce^{ax}$ when $n\neq a$,

and $\qquad=xe^{ax}+ce^{ax}$, when $n=a$.

Linear equation of the second order with constant coefficients.

255. The equation may be written

$$\frac{d^2y}{dx^2}+A\frac{dy}{dx}+By=X,$$

where A and B are constants and X is a constant or a function of x.

First suppose X to be 0. The equation is now

$$\frac{d^2y}{dx^2}+A\frac{dy}{dx}+By=0, \quad \text{or} \quad (D^2+AD+B)y=0.$$

Put D^2+AD+B into the form of factors $(D-a)(D-b)$. (Since a and b would be the roots of $D^2+AD+B=0$ if D were a symbol of quantity, they may be said to be the roots of the auxiliary equation $z^2+Az+B=0$.)

The equation is $(D-a)(D-b)y=0$.

$$\therefore\ y=(D-b)^{-1}(D-a)^{-1}0=(D-b)^{-1}ce^{ax}$$

$$=e^{bx}\left(\int e^{-bx}\,.\,ce^{ax}dx+c_1\right) \tag{1}$$

$$=e^{bx}\left(\frac{ce^{(a-b)x}}{a-b}+c_1\right), \text{ if } b\neq a,$$

$$=\frac{c}{a-b}e^{ax}+c_1e^{bx}.$$

Since $c/(a-b)$ may equal any constant, it may be represented by c.

$$\therefore\ y=ce^{ax}+c_1e^{bx}, \text{ if } b\neq a. \tag{2}$$

But if $b=a$, (1) becomes $e^{ax}\left(\int c\,dx+c_1\right)$.

$$\therefore\ y=e^{ax}(cx+c_1). \tag{3}$$

If a and b are imaginary, let them be $m+ni$, $m-ni$, where $i\equiv\sqrt{-1}$. Then (2) becomes

$$\begin{aligned} y&=ce^{(m+ni)x}+c_1e^{(m-ni)x}=e^{mx}(ce^{nix}+c_1e^{-nix})\\ &=e^{mx}[c(\cos nx+i\sin nx)+c_1(\cos nx-i\sin nx)],\ \S\ 200,\\ &=e^{mx}[(c+c_1)\cos nx+(ci-c_1i)\sin nx]. \end{aligned}$$

Since $c+c_1$ and $ci-c_1i$ may equal any two constants, they may be represented by c and c_1.

$$\therefore\ y=e^{mx}(c\cos nx+c_1\sin nx). \tag{4}$$

256. Hence to solve $D^2y+A\,Dy+By=0$ find the roots of $z^2+Az+B=0$. If the roots are unequal real numbers a and b the solution is

$$y=ce^{ax}+c_1e^{bx}.$$

If the roots are complex numbers $m+ni$, $m-ni$,

$$y=e^{mx}(c\cos nx+c_1\sin nx).$$

If the roots are equal numbers a, a,

$$y=e^{ax}(cx+c_1).$$

Ex. 1. $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}-10y=0.$ *Ans.* $y=ce^{2x}+c_1e^{-5x}.$

2. $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}=0.$ $y=c+c_1e^{3x}.$

3. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=0.$ $y=e^x(cx+c_1).$

4. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+10y=0.$ $y=e^{-x}(c\cos 3x+c_1\sin 3x).$

5. $\dfrac{d^2y}{dx^2}=a^2y.$ $y=ce^{ax}+c_1e^{-ax}$ or $=c\cosh ax+c_1\sinh ax.$

6. $\dfrac{d^2y}{dx^2}=-a^2y.$ $y=c\cos ax+c_1\sin ax.$

7. $(D^2+4D+5)y=0.$ $y=e^{-2x}(c\cos x+c_1\sin x).$

8. $(D^2+4D+4)y=0.$ $y=e^{-2x}(cx+c_1).$

9. $(D^2+4D+3)y=0.$ $y=ce^{-x}+c_1e^{-3x}.$

10. $(D^2+4D+2)y=0.$ $y=e^{-2x}(ce^{\sqrt{2}x}+c_1e^{-\sqrt{2}x}).$

11. $(6D^2-5D-6)y=0.$ $y=ce^{\frac{3}{2}x}+c_1e^{-\frac{2}{3}x}.$

257. The same method may be extended to higher orders of linear equations of the form

$$\frac{d^ny}{dx^n}+A\frac{d^{n-1}y}{dx^{n-1}}+\ldots+Ky=0,$$

the coefficients $A\ldots K$ being constants. For every distinct real root of the auxiliary equation,

$$z^n+Az^{n-1}+\ldots+K=0,$$

there will be a term of the form ce^{ax} in the solution. If a occur twice the corresponding term will be $e^{ax}(cx+c_1)$, and if it occur three times the corresponding term will be

$$e^{ax}(cx^2+c_1x+c_2),$$

and so on.

Corresponding to a pair of imaginary roots $m+ni$, $m-ni$, there will be a term $e^{mx}(c\cos nx+c_1\sin nx)$, and if the same pair occur twice the corresponding term in the solution will be

$$e^{mx}[(cx+c_1)\cos nx+(c_2x+c_3)\sin nx],$$

and so on.

Ex. 1. $(D-1)(D-3)^2y=0$. *Ans.* $y=ce^x+(c_1x^2+c_2x+c_3)e^{3x}$.

2. $(D^2+4)^2y=0$. *Ans.* $y=(c_1x+c_2)\cos 2x+(c_3x+c_4)\sin 2x$.

3. $(D^2-4)^2y=0$. $y=(c_1x+c_2)e^{2x}+(c_3x+c_4)e^{-2x}$.

4. $D^4y=a^4y$. $y=c_1e^{ax}+c_2e^{-ax}+c_3\cos ax+c_4\sin ax$.

5. $(D^2-4D+13)^2y=0$.

Ans. $y=e^{2x}[(c_1x+c_2)\cos 3x+(c_3x+c_4)\sin 3x]$.

258. Returning to the linear equation,

$$\frac{d^2y}{dx^2}+A\frac{dy}{dx}+By=X,$$

suppose now that X is not zero. The equation is the same as

$$(D-a)(D-b)y=X=0+X,$$
$$\therefore\ y=(D-b)^{-1}(D-a)^{-1}0+(D-b)^{-1}(D-a)^{-1}X.$$

The first term is the solution of the given equation when X is 0. This therefore forms a part of the required solution; it is called the *complementary function*, the remainder of the solution being called the *particular integral*. The complementary function will contain two arbitrary constants, and as the complete solution of an equation of the second order cannot contain more, we need not introduce constants in finding the particular integral. If they are introduced they will simply reproduce the complementary function.

Ex. 1. $(D^2-a^2)y=e^{ax}$. The c. f. is $ce^{ax}+ce^{-ax}$. The p. i. is

$$(D-a)^{-1}(D+a)^{-1}e^{ax}=(D-a)^{-1}e^{-ax}\int e^{ax}\,.\,e^{ax}dx$$

$$=(D-a)^{-1}e^{-ax}\,.\,\frac{e^{2ax}}{2a}=\frac{1}{2a}(D-a)^{-1}e^{ax}$$

$$=\frac{1}{2a}e^{ax}\int e^{-ax}e^{ax}dx=\frac{1}{2a}e^{ax}x.$$

Hence the complete solution is

$$y=ce^{ax}+c_1e^{-ax}+xe^{ax}/2a.$$

[$(D+a)^{-1}(D-a)^{-1}e^{ax}$ gives $\dfrac{1}{2a}e^{ax}x-\dfrac{1}{4a^2}e^{ax}$. The last term is included in the c. f.; hence the results are equivalent.]

2. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=e^{2x}$. *Ans.* $y=e^{-x}(cx+c_1)+\frac{1}{9}e^{2x}$.

3. $(D^2-1)y=5x+2$. $y=ce^x+c_1e^{-x}-5x-2$.

4. $(D-1)^2y=x$. $y=e^x(cx+c_1)+x+2$.

5. $(D-a)^2y=e^{ax}$. $y=e^{ax}(cx+c_1)+\frac{1}{2}x^2e^{ax}$.

6. $(D^2-4D+3)y=x$. $y=ce^x+c_1e^{3x}+\frac{1}{3}x+\frac{4}{9}$.

7. $(D^2-4D+3)y=xe^x$. $y=ce^x+c_1e^{3x}-\frac{1}{4}e^x(x^2+x)$.

8. $(D^2+a^2)y=e^{ax}$.

The p. i. $=(D+ai)^{-1}(D-ai)^{-1}e^{ax}$, $i\equiv\sqrt{-1}$,

$$=(D+ai)^{-1}\frac{e^{ax}}{a(1-i)}=\frac{e^{ax}}{a(1-i)\,.\,a(1+i)}=\frac{e^{ax}}{2a^2}.$$

$\therefore\ y=c\cos ax+c_1\sin ax+e^{ax}/2a^2$.

259. The last equation, $(D^2+a^2)y=e^{ax}$, may also be solved as follows:

Differentiating, $D(D^2+a^2)y=ae^{ax}$,

and multiplying the given equation by a and subtracting,

$$(D-a)(D^2+a^2)y=0,$$

a linear equation with the second member zero. The solution of this equation is

$$y=c\cos ax+c_1\sin ax+c_2e^{ax}.$$

The first two terms are the c. f. of the given equation; hence c_2e^{ax} is the p. i., where c_2 is to be determined so that c_2e^{ax} may satisfy the given equation. Substituting c_2e^{ax} for y in the given equation we find $c_2 = 1/2a^2$.

Ex. 1. $(D^2-1)y = x^2$.

Differentiating three times, $D^3(D^2-1)y = 0$. The c. f. is $(D^2-1)^{-1}0 = ce^x + c_1e^{-x}$. The p. i. is $D^{-3}0 = c_2x^2 + c_3x + c_4$. Substituting this for y in the given equation we find $c_2 = -1$, $c_3 = 0$, $c_4 = -2$.

$$\therefore\ y = ce^x + c_1e^{-x} - x^2 - 2.$$

The p. i. might have been found more quickly by treating $(D^2-1)^{-1}$ as if it were developable by the Binomial Theorem. Thus $(D^2-1)^{-1}x^2 = -(1-D^2)^{-1}x^2 = -(1+D^2+\dots)x^2 = -(x^2+2)$.

2. $\dfrac{d^2y}{dx^2} + a^2y = b \sin nx$. Differentiate twice, eliminate the right-hand member and show that

$$y = c \cos ax + c_1 \sin ax + \frac{b}{a^2-n^2} \sin nx.$$

If $a = n$, substitute $-bx/2n$ for $b/(a^2-n^2)$.

3. $(D^2-2D+5)y = 1$. *Ans.* $y = \frac{1}{5} + e^x(c \cos 2x + c_1 \sin 2x)$.

4. $(D-1)^2y = x^2$. $y = e^x(cx + c_1) + x^2 + 4x + 6$.

5. $(D^2-2D+5)y = \sin 2x$.

$$y = \tfrac{1}{17}(4 \cos 2x + \sin 2x) + e^x(c \cos 2x + c_1 \sin 2x).$$

Change of Variable.

260. Equations of the second order, like those of the first order, are sometimes made integrable by a change of variable.

Change of the dependent variable.

Ex. 1. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^3$.

Let $y = vx$, then $dy = x\,dv + v\,dx$, $d^2y = x\,d^2v + 2dv\,dx$.

Substituting, the equation becomes

$$x\frac{d^2v}{dx^2} + 3\frac{dv}{dx} = x,$$

whence (§ 251) $y=\frac{1}{6}x^3+cx+c_1x^{-1}$.

2. $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1$. Let $y^2=v$. *Ans.* $y^2=x^2+cx+c_1$.

3. $d^2y/dx^2=a^2x-b^2y$. Let $a^2x-b^2y=v$.

Ans. $b^2y=a^2x+c\cos bx+c_1\sin bx$.

261. Change of the independent variable. First substitute

$$\frac{dx\,d^2y-dy\,d^2x}{dx^3} \tag{1}$$

for d^2y/dx^2 (§ 70). The subsequent change will depend upon the quantity which is to be the independent variable. This may be y, or a third variable z, x or y being an assigned function of z. It must be remembered that the second differential of the independent variable $=0$.

Ex. 1. $$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\frac{a^2y}{x^2}=0. \tag{2}$$

Substituting (1) for d^2y/dx^2, (2) becomes

$$x^2\frac{dx\,d^2y-dy\,d^2x}{dx^3}+2x\frac{dy}{dx}+\frac{a^2y}{x^2}=0. \tag{3}$$

Let $x=1/z$ and take z as independent variable. Then $dx=-dz/z^2$, $d^2x=2dz^2/z^3$. Substituting in (3) we obtain

$$\frac{d^2y}{dz^2}+a^2y=0, \text{ whence } y=c\cos az+c_1\sin az.$$

$$\therefore\ y=c\cos\frac{a}{x}+c_1\sin\frac{a}{x}.$$

2. $\frac{d^2y}{dx^2}+\frac{2x}{1+x^2}\frac{dy}{dx}+\frac{y}{(1+x^2)^2}=0$. Let $x=\tan z$.

Ans. $y=(c+c_1x)/\sqrt{1+x^2}$.

3. $\frac{d^2y}{dx^2}-x\left(\frac{dy}{dx}\right)^3+e^y\left(\frac{dy}{dx}\right)^3=0$. Make y the independent variable.

The equation becomes $\frac{d^2x}{dy^2}+x=e^y$, whence

$$x=c\cos y+c_1\sin y+\tfrac{1}{2}e^y.$$

4. $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2$. Let $x=\sin z$.

Ans. $y=(\sin^{-1}x)^2+c\sin^{-1}x+c_1$.

5. $(1-y^2)\frac{d^2y}{dx^2}+y\left(\frac{dy}{dx}\right)^2-x\left(\frac{dy}{dx}\right)^3=0$. Let $y=\sin z$, z to be the independent variable. *Ans.* $x=cy+c_1\sqrt{1-y^2}$.

6. Show that the "homogeneous linear equation" of the second order

$$x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=f(x)$$

becomes

$$\frac{d^2y}{dz^2}+(a-1)\frac{dy}{dz}+by=f(e^z)$$

(a linear equation with constant coefficients) by the substitution $x=e^z$.

7. $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y=0$. *Ans.* $y=cx+\frac{c_1}{x^2}$.

8. $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=\log x$. $y=(cx+1)\log x+c_1x+2$.

9. $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+4y=x^2$.

$y=c\cos(\log x^2)+c_1\sin(\log x^2)+\frac{1}{5}x^2$.

10. $(2+x)^2\frac{d^2y}{dx^2}+3(2+x)\frac{dy}{dx}+y=0$. Let $2+x=e^z$.

Ans. $y=(2+x)^{-1}[c\log(2+x)+c_1]$.

EXAMPLES.

1. Find the curve in which the radius of curvature R is equal to and in the same direction as the normal N.

$$R=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{(1+p^2)^{\frac{3}{2}}}{p\frac{dp}{dy}},\quad\text{and}\quad N=y\frac{ds}{dx}=y(1+p^2)^{\frac{1}{2}}.$$

Ans. The catenary $y=c\cosh(x+c_1)/c$.

2. Find the curve in which $R=-N$.

Ans. The circle $(x+c_1)^2+y^2=c^2$.

3. Find the curve in which $R=e^x$, it being given that $p=0$ when $x=\infty$. *Ans.* $y=c-\sec^{-1}e^x$.

4. An elastic string (or spiral spring) is fixed at one end and hangs vertically. A weight is attached at the lower end and descends a distance a to a position of equilibrium O. It is then pushed down a further distance $b(<a)$ and released. To find x, the distance of the weight from O, in terms of the time t from the instant of release.

It is assumed that the weight of the string is negligible, and that tension is proportional to extension (Hooke's Law). Hence, since mass × acceleration = force, the differential equation of the motion of the weight is

$$m\frac{d^2x}{dt^2} = mg - mg\frac{a+x}{a}, \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{g}{a}x. \tag{1}$$

The solution is

$$x = c\cos\sqrt{g/a}\,t + c_1\sin\sqrt{g/a}\,t.$$

When $t=0$, $x=b$ and dx/dt (the velocity) $=0$. Hence $c=b$ and $c_1=0$.

$$\therefore\ x = b\cos\sqrt{g/a}\,t,$$

the required result. The values of x are repeated in magnitude and sign after an interval T if $\sqrt{g/a}(t+T) = \sqrt{g/a}\,t + 2\pi$, i.e., if $T = 2\pi\sqrt{a/g}$. The motion of which (1) is the differential equation is therefore oscillatory (a simple harmonic motion) and the periodic time $= 2\pi\sqrt{a/g}$. Notice that the total extension of the string at time t is

$$a + b\cos\sqrt{g/a}\,t.$$

5. A fine chain of length $2a$ is placed over a smooth nail, the difference of the lengths on the two sides being $2b$. Find x, the length of the longer side, in terms of the time t from the instant of release.

The whole chain is moved by the weight of the difference of the two sides; hence, m being the mass per unit length, the differential equation of motion is

$$2am\frac{d^2x}{dt^2} = mg[x-(2a-x)], \quad \text{or} \quad \frac{d^2(x-a)}{dt^2} = \frac{g}{a}(x-a).$$

Ans. $x = a + b\cosh\sqrt{g/a}\,t.$

Show also that the chain will leave the nail in time
$\sqrt{a/g}\cosh^{-1}(a/b)$ and with velocity $\sqrt{g(a^2-b^2)/a}$.

Fig. 134.

6. **Dampel vibrations.** The differential equation of simple harmonic motion is (as in Ex. 4) $d^2x/dt^2 = -a^2x$. If a frictional or other resistance proportional to the speed is applied, the equation is (say)

$$\frac{d^2y}{dt^2} = -a^2x - 2m\frac{dx}{dt},$$

where m is a positive constant. If $m < a$ the solution is

$$x = e^{-mt}(c\cos nt + c_1 \sin nt), \quad \text{or} \quad = Ae^{-mt}\sin(nt+B),$$

where $n = \sqrt{a^2-m^2}$, and A and B are arbitrary constants. For $A=1$, $B=0$, $m=\cdot3$, and $a=1\cdot6$,

$$x = e^{-\cdot3t}\sin 1\cdot57t, \quad \text{(Fig. 134).}$$

If m is increased the waves become longer and flatter, and when $m=a$ the solution of the equation is

$$x = e^{-1\cdot6t}(cx + c_1),$$

which for $c=1$, $c_1=0$, is the dotted curve of the figure, the curve being asymptotic to the t-axis.

7. Find the distance x which a body of mass m falls from rest in time t, assuming that gravity is constant, and that the resistance of the atmosphere varies as the square of the velocity.

Let the resistance be k when the velocity $=1$. Then

$$m\frac{d^2x}{dt^2} = mg - k\left(\frac{dx}{dt}\right)^2. \qquad Ans.\ x = \frac{m}{k}\log\cosh\sqrt{\frac{gk}{m}}\,t.$$

8. A uniform beam is fixed horizontally at one end, loaded with a weight w per unit length, and subjected at C, the centre of the free end, to a vertical force P and a horizontal tensile force Q. The origin being at C and the x-axis horizontal, the equation of the elastic curve is

$$EI\frac{d^2y}{dx^2} - Qy = -Px - \tfrac{1}{2}wx^2,$$

where E and I are constants depending upon the material and form of the beam. Write the equation in the form

$$(D^2-a^2)y=-bx-fx^2,$$

where $a\equiv\sqrt{\frac{Q}{EI}}$, $b\equiv\frac{P}{EI}$, $f\equiv\frac{w}{2EI}$.

The c. f. $=ce^{ax}+c_1e^{-ax}$. The p. i. can be found by the method of § 259, but more quickly by treating D^2-a^2 as if it were a symbol of quantity. Thus

$$-(D^2-a^2)^{-1}(bx+fx^2)=\frac{1}{a^2}\left(1-\frac{D^2}{a^2}\right)^{-1}(bx+fx^2)$$

$$=\frac{1}{a^2}\left(1+\frac{D^2}{a^2}+\ldots\right)(bx+fx^2)=\frac{1}{a^2}\left(bx+fx^2+\frac{2f}{a^2}\right).$$

$$\therefore\ y=ce^{ax}+c_1e^{-ax}+\frac{bx}{a^2}+\frac{fx^2}{a^2}+\frac{2f}{a^4}.$$

To determine c and c_1, $x=0$ when $y=0$,

$$\therefore\ 0=c+c_1+2f/a^4.$$

Also, since the tangent at the fixed end is horizontal, $dy/dx=0$ when $x=$ the length l, very nearly.

$$\therefore\ 0=a(ce^{al}-c_1e^{-al})+b/a^2+2fl/a^2.$$

These two equations may be solved for c and c_1.

9. If Q (Ex. 8) acts in the opposite direction show that

$$y=c\cos ax+c_1\sin ax+\frac{bx}{a^2}+\frac{fx^2}{a^2}-\frac{2f}{a^4}.$$

10. **Curve of pursuit.** To find the path of a dog which runs to overtake his master, both moving with uniform speed, and the latter in a straight line.

Take this line for y-axis. When the dog is at (x, y) on the curve, the distance of the man from the origin is $y-px$ (the intercept of the tangent on the y-axis), and by supposition,

$$\frac{d(y-px)}{dt}=k\frac{ds}{dt},$$

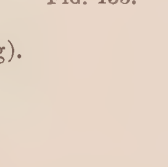

FIG. 135.

where k is a constant (speed of man ÷ speed of dog).

$$\therefore\ -x\,dp=k\sqrt{1+p^2}\,dx$$

is the differential equation of the curve. The solution is

$$2y = c_1 + \frac{cx^{1+k}}{1+k} - \frac{c^{-1}x^{1-k}}{1-k}, \quad \text{if } k \neq 1,$$

and

$$= c_1 + \tfrac{1}{2}cx^2 - c^{-1}\log x, \quad \text{if } k = 1.$$

The constants c and c_1 can be determined if the initial conditions are assigned.

If the man starts from O, the dog from any point A on the x-axis (see figure), and they meet at B, show that the length of the curve $= AC + CB$, where C is the middle point of OA.

APPENDIX.

Note A. Partial Fractions.

The algebraical sum * of certain fractions being given, it is required to find the fractions.

Case 1. When the factors of the denominator are all of the first degree and unequal.

Ex. $\dfrac{x^2+3x+1}{x(x-1)(x+2)}$.

The denominator indicates that this result may be obtained by the addition of three fractions whose denominators are x, $x-1$, $x+2$, respectively; our object then is to find the numerators. Call them A, B, C.†

$$\therefore \frac{x^2+3x+1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}. \qquad (1)$$

Clearing of fractions,

$$x^2+3x+1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1). \qquad (2)$$

It is to be noticed that (1), and $\therefore$ (2), is to be an identity and therefore true for all values of x. Now, if we give *any* three successive arbitrary values to x in (2), we shall obtain three equations by solving which, as simultaneous equations, the quantities A, B, C may be found. The arbitrary values should, however, be such as to render these three equations as easy of solution

* The sum is assumed to be a proper fraction (the numerator of lower dimensions than the denominator). If not, the fraction should be reduced to a mixed quantity.

† The A, B, C are assumed to be independent of x. If one of them contained x the sum would not be a proper fraction.

as possible. A little inspection will show that this will be accomplished by giving x successive values which make x, $x-1$, and $x+2$ equal to 0, i.e., by making $x=0$, 1, and -2. We thus get from (2),

$$\begin{aligned} &\text{when } x=0, && 1=A(-2), && \therefore A=-\tfrac{1}{2};\\ &\text{when } x=1, && 5=B(3), && \therefore B=\tfrac{5}{3};\\ &\text{when } x=-2, && -1=C(6), && \therefore C=-\tfrac{1}{6}. \end{aligned}$$

$$\therefore \frac{x^2+3x+1}{x(x-1)(x+2)}=\frac{-\frac{1}{2}}{x}+\frac{\frac{5}{3}}{x-1}+\frac{-\frac{1}{6}}{x+2}$$

$$=-\frac{1}{2x}+\frac{5}{3(x-1)}-\frac{1}{6(x+2)}.$$

Case 2. When the factors of the denominator are of the first degree, but two or more of them are equal.

Ex. 1. $\dfrac{1+3x}{x(x+1)^2}$.

The denominator shows that the partial fractions have denominators x and $(x+1)^2$ and (probably) $x+1$. We therefore assume

$$\frac{1+3x}{x(x+1)^2}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}.$$

$$\therefore 1+3x=A(x+1)^2+Bx(x+1)+Cx.$$

If $x=0$, $1=A$, $\therefore A=1$.

If $x=-1$, $-2=-C$, $\therefore C=2$,

and B may be found by giving *any* value other than 0 and -1 to x, e.g., if $x=1$ we have ($\because A=1$, and $C=2$),

$$4=1\times 2^2+B\times 2+2\times 1, \quad \therefore B=-1,$$

$$\therefore \frac{1+3x}{x(x+1)^2}=\frac{1}{x}-\frac{1}{x+1}+\frac{2}{(x+1)^2}.$$

2. $\dfrac{x+2}{(x+1)(x-1)^3}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}+\dfrac{D}{(x-1)^3}$.

$\therefore x+2=A(x-1)^3+B(x+1)(x-1)^2+C(x+1)(x-1)+D(x+1)$.

If $x=-1$, $1=A(-2)^3$, $\therefore A=-\frac{1}{8}$.

If $x=1$, $3=D(2)$, $\therefore D=\frac{3}{2}$.

To get B and C give x any two arbitrary values other than 1 and -1; thus (remembering that A and D are found) if $x=0$,

$$2=\tfrac{1}{8}+B-C+\tfrac{3}{2} \quad \text{or} \quad B-C=\tfrac{3}{8},$$

and if $x=3$, $2B+C=0$; hence from these two equations

$$B=\tfrac{1}{8}, \quad C=-\tfrac{1}{4}.$$

$$\therefore \frac{x+2}{(x+1)(x-1)^3}=-\frac{1}{8(x+1)}+\frac{1}{8(x-1)}-\frac{1}{4(x-1)^2}+\frac{3}{2(x-1)^3}.$$

Case 3. When the denominator contains a quadratic factor which cannot be conveniently factorized. We now assume the numerator of the fraction with a quadratic denominator to be of the form $Ax+B$. This is equivalent to assuming two fractions with denominators of the first degree and constant numerators.

Ex. $\dfrac{1+x}{x(1+x^2)}=\dfrac{Ax+B}{1+x^2}+\dfrac{C}{x}$. $\quad\therefore\ 1+x=Ax^2+Bx+C(1+x^2)$.

If $x=0$, $\quad 1=C$, $\quad\therefore\ C=1$.

If $x=1$, $\quad 2=A+B+2$, $\quad\therefore\ A+B=0$,

If $x=-1$, $0=A-B+2$, $\quad\therefore\ A-B=-2$,

$\therefore\ A=-1$, $\quad B=1$.

$$\therefore \frac{1+x}{x(1+x^2)}=\frac{-x+1}{1+x^2}+\frac{1}{x}.$$

If the given denominator had contained the square of $1+x^2$, we should have assumed an additional term $\dfrac{Dx+E}{(1+x^2)^2}$.

Besides the methods explained in the above examples others may sometimes be employed with advantage. For instance, in the last example

$$1+x=Ax^2+Bx+C(1+x^2).$$

Since the left- and right-hand sides are to be identical, the coefficients of like powers of x on the two sides must be equal; we $\therefore$ have $1=C$, $1=B$, $0=A+C$, which give the same results as before.

Note B. Curve Tracing.

1. In order to trace a curve accurately from its equation we must be able to express one of the coördinates in terms of the other, or both in terms of a third variable. When the rectangular equation contains terms of two degrees only, we may substitute mx for y and solve for x, and in this way obtain both x and y in terms of m. See foot-note, p. 53.

2. The following suggestions and remarks may be found useful in curve tracing, in order to shorten or verify the work.

(I) Examine the equation for *symmetry*. When the equation remains unchanged if $-y$ is substituted for y the curve is symmetrical with reference to the line $y=0$ (the x-axis), for if the coördinates (a, b) satisfy the equation, $(a, -b)$ will also satisfy it. This will always be the case if the equation contains only even powers of y. Similarly the curve is symmetrical with reference to the line $x=0$ (the y-axis) if its equation is not altered when x is changed into $-x$.

If the equation is unaltered by changing x into $-x$ and y into $-y$ at the same time, every line drawn through the origin and terminated by the curve is bisected by the origin; for if (a, b) satisfy the equation, $(-a, -b)$ also satisfy it and the origin is the middle point of the line joining these points. The origin is then called a *centre;* e.g., in the curves $y=x^3$, $y=\sin x$, etc.

The curve is symmetrical with reference to the line $y=x$ if the equation is unaltered when x is changed into y and y into x, e.g., $x^3+y^3=3axy$ (Fig. 28); and it is symmetrical with reference to the line $y=-x$ if we can change y into $-x$ and x into $-y$ without altering the equation, e.g., in $x^3-y^3=3axy$.

If in polar equations the substitution of $-\theta$ for θ does not alter the equation, the curve is symmetrical with reference to the initial line (e.g., in Figs. 88, 89, 90, 92); and if we may at the same time change r into $-r$ and θ into $-\theta$ without altering the equation, the curve is symmetrical with

reference to a line through the origin perpendicular to the initial line (e.g., in Figs. 84, 85, 98). The origin is a centre when we can change r into $-r$ without altering the equation (e.g., in Figs. 86, 98).

(II) Find the tangents at the origin (if the origin lie on the curve) and the shape of the curve near the origin (§§ 3 and 4 below); also, if possible, the points of intersection of the curve and the axes, and the directions of the tangents at these points; the points where the coördinates are maxima or minima; the points of inflexion; the asymptotes rectilinear or curvilinear, etc.

(III) No straight line can meet a curve of the nth degree in more than n points, and therefore no tangent in more than $n-2$ points besides the point of contact, no asymptote in more than $n-2$ points at a finite distance and no line parallel to an asymptote in more than $n-1$ points at a finite distance, no line through a double point in more than $n-2$ other points, etc.

3. The work of tracing a curve from its equation is often considerably lightened by obtaining a preliminary idea of the shape of the curve at certain points.

When the origin is a point on a curve we can find the shape of the curve very near that point by expanding y into a series of ascending powers of x. Thus in Fig. 32, $y=\pm x(1+2x)^{\frac{1}{2}}$, and taking first the + sign we have by the Binomial Theorem,

$$y=x(1+x-\dots), \quad \text{or} \quad y=x+x^2-\dots$$

The term x^2 shows that when x is very small (and $\therefore$ the third and higher powers of x may be neglected) the curve lies above its tangent $y=x$ both when x is + and when x is −; in fact the curve is, for points near the origin on the branch touching $y=x$, shaped nearly like the parabola $y=x+x^2$. Similarly on the other branch $y=-x-x^2+\dots$; hence this branch lies below the tangent on both sides of the origin.

Similarly we may show that in Fig. 34 the curve near the origin is shaped nearly like the parabolas $y=x^2$, $y=-x^2$.

4. When it is not convenient or possible to express one coördinate in terms of the other we may proceed as in the following examples:

Ex. 1 In the curve $a^2(y-x)(y+x)=-(y^2+x^2)^2$, Fig. 27, considering first the branch which touches $y-x=0$ (§ 52) we divide by $a^2(y+x)$ and write the equation in the form

$$y=x-\frac{(y^2+x^2)^2}{a^2(y+x)}. \tag{1}$$

For points near the origin on the branch in question y is very nearly equal to x, and the fraction in (1) must be very small; we shall get an approximation to its value by substituting x for y; this gives

$$y=x-2x^3/a^2,$$

which shows that the curve lies below the tangent when x is + and above it when x is $-$. For the other branch we write the equation in the form

$$y=-x-\frac{(y^2+x^2)^2}{a^2(y+x)}, \tag{2}$$

and remembering that y is nearly equal to $-x$ we substitute $-x$ for y in the fraction and get

$$y=-x+2x^3/a^2,$$

showing that the curve lies above the tangent when x is + and below when x is $-$.

2. In the curve $3axy=x^3+y^3$, Fig. 28, the tangents at the origin are $y=0$ and $x=0$. Writing the equation in the form

$$y=\frac{x^3+y^3}{3ax}$$

we observe that on the branch which touches $y=0$ (the x-axis) y is nearly 0 near the origin, and substituting this for y in the

fraction gives $y=x^2/3a$ for the approximate form of the curve. For the other branch

$$x=\frac{x^3+y^3}{3ay},$$

and writing 0 for x in the second member we get $x=y^2/3a$ for the required approximation. Thus the curve is shaped near the origin like a pair of parabolas.

3. Find the approximations to the three branches of the curve $ay(y-\sqrt{3}x)(y+\sqrt{3}x)=x^4$, Fig. 36, near the origin.

Ans. $6a(y-\sqrt{3}x)=x^2$, $6a(y+\sqrt{3}x)=x^2$, $3ay=-x^2$.

4. Also of $ay^2(y-x)(y+x)=x^5$, Fig. 37.

Ans. $ay^2=-x^3$, $2a(y-x)=x^2$, $2a(y+x)=-x^2$.

5. Show from these approximations that the radii of curvature at the origin are $\frac{3}{2}a$ and $\pm 24a$ in Fig. 36, and 0, $\pm 2\sqrt{2}a$ in Fig. 37. (Cf. § 88, Ex. 2.)

5. The asymptotes of a curve may be obtained by expanding y into a series of descending powers of x (see § 57). When it is impossible or difficult to express one of the coördinates in terms of the other we may proceed in a manner similar to that of § 4 above, beginning, however, with the terms of the highest degree instead of those of the lowest. (See § 59.)

Ex. 1. $x^3+y^3=3axy$, Fig. 28. Here $x+y$ is a factor of the terms of the highest degree, and we may write the equation in the form

$$y=-x+\frac{3axy}{x^2-xy+y^2}. \qquad (1)$$

Now the infinite branch is in the direction of the line $y=-x$, and therefore when x is very large, y is nearly equal to $-x$; hence we shall get an approximation to the fraction in (1) by substituting $-x$ for y; this gives

$$y=-x-a,$$

which is the nearest linear approximation to the curve, and is therefore the equation of the asymptote. Writing $-x-a$ for y in the fraction will give a second approximation, viz.,

$$y=-x-a+\frac{a^3}{3x^2},$$

from which it appears that the curve lies above the asymptote whether x is $+$ or $-$.

2. Find by this method the asymptotes of the following curves:

(1) $x^3(y-x)=a(y^3+x^3)$. *Ans.* $y=x+2a$.
(2) $xy^2(y-x)=y^3-2x^2y+x^2$. $x=1,\quad y=1,\quad y=x-1$.
(3) $(x+2y)(x-y)^2=6a^2(x+y)$. $x+2y=0,\quad x-y=\pm 2a$.

EXAMPLES.

1. Trace the following curves:*

(1) $y=x(x^2-1)$,
(2) $y(x^2-1)=x$,
(3) $y(1+x^2)=x$,
(4) $y^2=x^3(x+1)$,
(5) $y^3=x^3(x-1)$,
(6) $y^2=x^2(x-1)$,
(7) $x^3-y^3=3axy$,
(8) $x^4+y^4=a^2xy$,
(9) $x^5+y^5=2a^3xy$,
(10) $x^5+y^5=ax^4$,
(11) $x^3+y^3=a^3$,
(12) $x(y-x)=ay^2$,
(13) $x(y-x)^2=y^4$,
(14) $a^2y(x+y)=x^4$.

2. Trace the following polar curves:

(1) $r=a\sin 2\theta$,
(2) $r\sin 2\theta=a$,
(3) $r=a\sin 3\theta$,
(4) $r\sin 3\theta=a$,
(5) $r^2=a^2\sin 3\theta$,
(6) $r=a\tan\theta$,
(7) $r^2=a^2\theta$,
(8) $r=a\theta^2$,
(9) $r\theta^2=a$,
(10) $r(1+\theta)=a\theta$,
(11) $r(\theta^2-1)=a\theta$,
(12) $r\theta^2=a(\theta^2-1)$,
(13) $r(\theta^2+1)=a\theta^2$,
(14) $r\theta=\tan\theta$.

Note C. Hyperbolic Functions.

(For definitions and graphs of these functions see Ch VIII.)

1. The relations connecting the hyperbolic functions are similar to those connecting circular (trigonometrical) functions, and are easily proved by ordinary algebra, etc. Some of them are as follows:

$$\cosh 0=1,\quad \sinh 0=0,\quad \tanh 0=0, \text{ etc.}$$

$\cosh(-x)=\cosh x$, $\sinh(-x)=-\sinh x$, $\tanh(-x)=-\tanh x$, etc.

* Some of these examples are taken from Frost's *Curve Tracing*, to which the student is referred for further information on this subject.

$$\cosh^2 x - \sinh^2 x = 1, \quad \operatorname{sech}^2 x = 1 - \tanh^2 x, \quad \operatorname{cosech}^2 x = \coth^2 x - 1.$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y,$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$

$$\cosh x + \cosh y = 2 \cosh \tfrac{1}{2}(x+y) \cosh \tfrac{1}{2}(x-y),$$

$$\cosh x - \cosh y = 2 \sinh \tfrac{1}{2}(x+y) \sinh \tfrac{1}{2}(x-y),$$

$$\sinh x + \sinh y = 2 \sinh \tfrac{1}{2}(x+y) \cosh \tfrac{1}{2}(x-y),$$

$$\sinh x - \sinh y = 2 \cosh \tfrac{1}{2}(x+y) \sinh \tfrac{1}{2}(x-y),$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$

$$= 2\cosh^2 x - 1 = 1 + 2\sinh^2 x,$$

$$\sinh 2x = 2 \sinh x \cosh x,$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y},$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}.$$

2. The differentials, integrals, etc., are as follows:

$$d \sinh x = \cosh x \, dx, \qquad \int \cosh x \, dx = \sinh x.$$

$$d \cosh x = \sinh x \, dx, \qquad \int \sinh x \, dx = \cosh x.$$

$$d \tanh x = \operatorname{sech}^2 x \, dx, \qquad \int \operatorname{sech}^2 x \, dx = \tanh x.$$

$$d \coth x = -\operatorname{cosech}^2 x \, dx, \qquad \int \operatorname{cosech}^2 x \, dx = -\coth x.$$

$$d \operatorname{sech} x = -\operatorname{sech} x \tanh x \, dx, \qquad \int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x.$$

$$d \operatorname{cosech} x = -\operatorname{cosech} x \coth x \, dx, \qquad \int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x.$$

$$d \sinh^{-1} \frac{x}{a} = \frac{dx}{\sqrt{x^2 + a^2}}, \qquad \int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} = \log\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right).$$

$$d\cosh^{-1}\frac{x}{a}=\frac{dx}{\sqrt{x^2-a^2}},\quad \int\frac{dx}{\sqrt{x^2-a^2}}=\cosh^{-1}\frac{x}{a}=\log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right).$$

$$d\tanh^{-1}\frac{x}{a}=\frac{a\,dx}{a^2-x^2},\ x|<|a,$$

$$\int\frac{dx}{a^2-x^2}(x|<|a)=\frac{1}{a}\tanh^{-1}\frac{x}{a}=\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right).$$

$$d\coth^{-1}\frac{x}{a}=\frac{a\,dx}{a^2-x^2},\quad x|>|a,$$

$$\int\frac{dx}{a^2-x^2}(x|>|a)=\frac{1}{a}\coth^{-1}\frac{x}{a}=\frac{1}{2a}\log\left(\frac{x+a}{x-a}\right).$$

$$d\,\mathrm{sech}^{-1}\frac{x}{a}=-\frac{a\,dx}{x\sqrt{a^2-x^2}},$$

$$\int\frac{dx}{x\sqrt{a^2-x^2}}=-\frac{1}{a}\,\mathrm{sech}^{-1}\frac{x}{a}=\frac{1}{a}\log\left(\frac{x}{a+\sqrt{a^2-x^2}}\right).$$

$$d\,\mathrm{cosech}^{-1}\frac{x}{a}=-\frac{a\,dx}{x\sqrt{a^2+x^2}},$$

$$\int\frac{dx}{x\sqrt{a^2+x^2}}=-\frac{1}{a}\,\mathrm{cosech}^{-1}\frac{x}{a}=\frac{1}{a}\log\left(\frac{x}{a+\sqrt{a\ +x}}\right).$$

$$\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$$

$$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$$

$$\tanh^{-1}x=x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots$$

$$\sinh^{-1}x=x-\frac{1}{2}\frac{x^3}{3}+\frac{1\,.\,3}{2\,.\,4}\frac{x^5}{5}-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{x^7}{7}+\ldots$$

If $i\equiv\sqrt{-1}$, $\cos ix=\cosh x$, $\sin ix=i\sinh x$, $\cosh ix=\cos x$. $\sinh ix=i\sin x$.

3. At any point of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (Fig. 136) we may put $x=a\cos u$, $y=b\sin u$, since $\cos^2 u+\sin^2 u=1$. In this

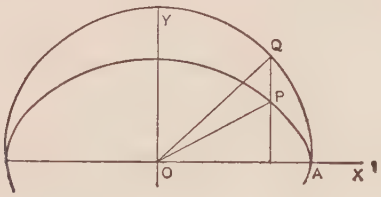

FIG. 136.

case $u=2$ area AOP/ab (see Ex. 14, p. 162); it also = the eccentric angle AOQ.

At any point of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (Fig. 137) we may put $x=a\cosh u$, $y=b\sinh u$, since $\cosh^2 u-\sinh^2 u=1$. In this case $u=2$ area $AOP/ab=\log\left(\frac{x}{a}+\frac{y}{b}\right)$, (see Ex. 14, p. 140).

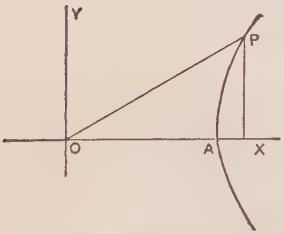

FIG. 137.

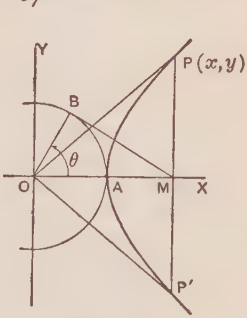

FIG. 138.

If $b=a$ the ellipse becomes the circle $x^2+y^2=a^2$, and the hyperbola the equilateral hyperbola $x^2-y^2=a^2$. Also u is in both cases the measure of the area of the sector AOP when $\frac{1}{2}a^2$ is taken as unit area. The circular and hyperbolic

functions may be defined in terms of u, and corresponding to

$$y/a = \sin u, \quad x/a = \cos u, \quad y/x = \tan u, \quad \text{etc.},$$

for the circle, we have

$$y/a = \sinh u, \quad x/a = \cosh u, \quad y/x = \tanh u, \quad \text{etc.},$$

for the equilateral hyperbola.

4. **Gudermannian.** If $z = \log \tan (\frac{1}{4}\pi + \frac{1}{2}\theta)$ or $\log (\sec \theta + \tan \theta)$, θ is called the gudermannian of z (gd z) and $z = \text{gd}^{-1}\theta$.

Since $e^z = \sec \theta + \tan \theta$, $\therefore e^{-z} = \sec \theta - \tan \theta$. Hence $\cosh z = \sec \theta$, $\sinh z = \tan \theta$, $\tanh z = \sin \theta$, etc. Thus if θ is tabulated for values of z the hyperbolic functions may be obtained from a table of circular functions.

Differentiating one of the relations connecting θ and z, we obtain $d\theta = \text{sech } z\, dz$, or $dz = \sec \theta\, d\theta$.

$$\therefore d(\text{gd } z) = \text{sech } z\, dz, \quad \text{and} \quad d(\text{gd}^{-1}\theta) = \sec \theta\, d\theta.$$

The inverse gudermannian is also written $\lambda(\theta)$ and called the lambda function, i.e.,

$$\lambda(\theta) = \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\theta) = \log (\sec \theta + \tan \theta).$$

Ex. Show that $\tanh \frac{1}{2}z = \tan \frac{1}{2}\theta$.

5. In the equilateral hyperbola $x^2 - y^2 = a^2$, Fig. 138, let u be, as in § 3 above, the area of the sector AOP in terms of $\frac{1}{2}a^2$ as unit area. From the foot M of the ordinate MP draw MB tangent to the circle $x^2 + y^2 = a^2$. Then $x/a = \cosh u$ and also $= \sec \theta$. $\therefore \theta$ is the gudermannian of u. It may also be proved that (1) $MB = y$, (2) $\tanh u = \tan AOP = \sin \theta$, (3) the line through O parallel to BP bisects both sectors AOB, AOP, and the chord AP.

Note D. Mechanical Integration.

1. **Sign of an area.** Let a straight line AB of constant length move in a plane to any other position $A'B'$, thus describing or sweeping out an area. Let it be agreed that any portion of AB describes a positive or a negative area according as, when viewed from A, it moves toward the left or the right. Thus the whole area is + in Fig. 139, − in Fig. 140, while in Fig. 141, BOB' is + and AOA' is −.

Fig. 139.

Fig. 140.

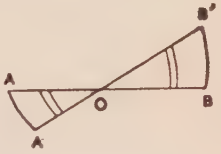

Fig. 141.

2. **Measurement of the area.** AB can be moved to any other position $A'B'$ (Fig. 142) by (1) a translation to $A'D$, during which the points in AB describe straight lines, and (2) a rotation about A', during which the points describe arcs of circles. The middle point M of AB moves first to F and then to M'. Take ME perpendicular to AB. The area of the parallelogram $AD = AB \,.\, ME$, the area of the sector $DA'B' = A'D \,.\, FM'$; hence the whole area $= AB(ME + FM')$, i.e., $AB \times$ the total normal displacement of its middle point.

Suppose a wheel to be attached to AB at M with its axis in the direction AB, and that suitable graduations record the number of revolutions and parts of a revolution which the wheel makes. Let n be this number, i.e., the change of reading of the recording circles between the time of starting and any subsequent time. Let $c =$ the length of the circumference of the wheel; then cn is the distance

rolled through by a point in the circumference of the wheel. Take b for the length of AB. During the motion of translation the wheel rolls over ME and slides through EF, during the rotation it rolls over FM'. Hence the total normal displacement of $M = cn$, and the total area described by $AB = bcn$.

If, as in Figs. 139, 140, 141, A and B describe curves, imagine the motion to be a combined translation and rotation with infinitesimal displacements, any of which may be negative as well as positive. Then the *rate* at which the area is described is $b \times$ rate of normal displacement of M,

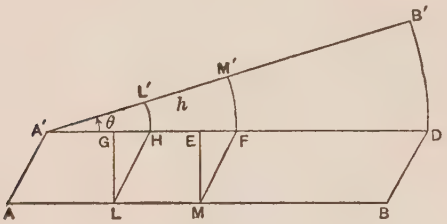

FIG. 142.

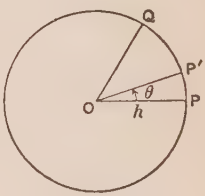

FIG. 143.

and hence the total resultant area $= b \times$ total normal displacement of $M = bcn$.

3. Consider now the effect of putting the wheel at any point L in AB, Fig. 142. The distance rolled over by the wheel is now $LG + HL'$, and hence the normal displacement of $M = cn + FM' - HL' = cn + (A'F - A'H)\theta = cn + h\theta$, if $LM = h$. In a circle (Fig. 143) of radius h draw OP, OP' parallel to AB, $A'B'$. Then $h\theta = PP'$. Hence the area described by $AB = b(cn + PP')$, and if AB moves to any new position to which OQ is parallel, the resultant area swept out $= b(cn + PQ)$. If AB moves about, turns back, and finally returns to its first position, Q returns to P and the resultant area $= bcn$, as if the wheel were at M. But if AB makes a complete revolution and returns to its first position the resultant area $= b(cn + 2\pi h)$.

4. Closed curves. Let a straight line move so that its extremities describe any closed curves. Then in all cases the area swept out by the line will be equal to the arithmetical difference of the areas of the curves described by its extremities.

When the areas are without one another, one will be described on the whole positively and the other on the whole negatively, while the area between them, if swept out at all, will be swept both positively and negatively. When they intersect, the common portion, in so far as it is swept at all, will be swept both positively and negatively; the rest as before. When one curve lies entirely inside the other, the portion of the former which is swept at all will be swept both positively and negatively.

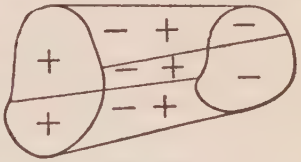

FIG. 144.

5. Amsler's polar planimeter consists essentially of two bars, CA, AB, hinged at A, a recording wheel being attached to AB at any point L. C is fixed while B is moved round a curve. But if A is constrained to move along any line—whether straight or curved—without enclosing any area, the area of the curve traced out by B is equal to the resultant area swept out by AB, and hence will be bcn if AB returns to its starting place without making a complete revolution. But if C lies inside the curve described by B, AB makes a complete revolution, and the area of the curve described by B

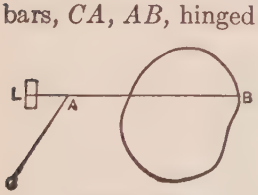

FIG. 145.

$$= b(cn + 2\pi h) + \text{circle described by } A$$

$$= b(cn + 2\pi h) + \pi a^2 = bc\left(n + \frac{2\pi bh + \pi a^2}{bc}\right).$$

The second term in the parentheses is constant (independent of n) and should be engraved on, or otherwise supplied with, the instrument. This number is then to be looked upon as a correction to n when the planimeter makes a complete revolution. $2\pi bh+\pi a^2$ is evidently the area of the circle described by B when n remains $=0$, i.e., when the instrument is set so that the wheel slides without rolling, or when the perpendicular from C on AB passes through the wheel.

By sliding the bar AB through a sleeve to which the hinge and the wheel are attached, its length may be altered and the instrument adapted to different units. Thus if the circumference of the wheel $=c$ centimetres, and b is taken $=100/c$, $bcn=100n$, and hence the area is found in square centimetres by multiplying n by 100. Similarly if the circumference of the wheel is c inches and b is taken $=10/c$, the area $bcn=10n$ square inches.

6. As we proceed from B to C by way of P, Fig. 146, x changes from OD to OE, and $\int y\,dx$ is the area $DBPCE$; but if we proceed from C to B by way of P', each element of area such as $y\,dx$ is negative since dx is negative, and hence $\int y\,dx$ is the area $CEDB$, but is negative. Hence if we sum the elements such as $y\,dx$ in the order of proceeding clockwise round the curve, the result $=DBPCE-DBP'CE=BPCP'$, the area of the curve. Let $A=$this area, $M=$the sum of the moments of the elements of the area with respect to OX, $I=$the moment of inertia of the area with respect to OX. Then

$$A=\int y\,dx, \quad M=\int y\,dx\,.\,\frac{y}{2}=\frac{1}{2}\int y^2\,dx, \quad I=\int y\,dx\,.\,\frac{y^2}{3}=\frac{1}{3}\int y^3\,dx.$$

7. **Amsler's mechanical integrator.** In this instrument one end of a sweeping bar FP traces a closed curve, while the other end is constrained to describe a straight line OX. Hence this part of the instrument is virtually a planimeter, and the area $A=bc_1n_1$, where b is the length of FP, c_1 the cir-

cumference of the wheel W_1 which FP carries, and n_1 the change of reading of this wheel when the circuit of the curve has been made. FP also turns two arcs of centre F and radii $2a$ and $3a$, which turn circles each of radius a, the circles carrying wheels W_2 and W_3. The three wheels roll simultaneously on the plane containing the diagram to be integrated. When the axis of W_1 makes an angle θ with OX

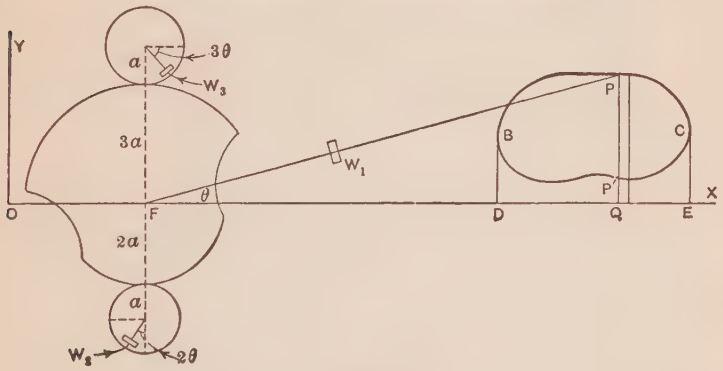

FIG. 146.

the axes of W_2 and W_3 make angles $\frac{1}{2}\pi-2\theta$ and 3θ, respectively, with the same line.

For y substitute $b \sin\theta$. Then

$$A=b\int \sin\theta\, dx, \qquad M=\tfrac{1}{2}b^2\int \sin^2\theta\, dx, \qquad I=\tfrac{1}{3}b^3\int \sin^3\theta\, dx.$$

Now $\quad 2\sin\theta=1-\cos 2\theta=1-\sin(\tfrac{1}{2}\pi-2\theta),$

and $\sin 3\theta=3\sin\theta-4\sin^3\theta$, or $\sin^3\theta=\tfrac{3}{4}\sin\theta-\tfrac{1}{4}\sin 3\theta.$

$$\therefore\ M=\tfrac{1}{4}b^2\int[1-\sin(\tfrac{1}{2}\pi-2\theta)]dx=-\tfrac{1}{4}b\int \sin(\tfrac{1}{2}\pi-2\theta)dx,$$

since $\int dx=0$ for the complete circuit of the curve.

Also,
$$I=\tfrac{1}{3}b^3\int(\tfrac{3}{4}\sin\theta-\tfrac{1}{4}\sin 3\theta)dx$$
$$=\tfrac{1}{4}b^3\int \sin\theta\, dx-\tfrac{1}{12}b^2\int \sin 3\theta\, dx.$$

But $A = bc_1n_1$, $\therefore \int \sin\theta\, dx = c_1n_1$, i.e., when the axis of a wheel makes an angle θ with OX, $\int \sin\theta\, dx = c_1n_1$. But the axes of the other wheels make angles $\frac{1}{2}\pi - 2\theta$ and 3θ with OX.

$$\therefore \int \sin(\tfrac{1}{2}\pi - 2\theta)dx = c_2n_2, \quad \text{and} \quad \int \sin 3\theta\, dx = c_3n_3.$$

$$\therefore A = bc_1n_1, \quad M = \tfrac{1}{4}b^2c_2n_2, \quad I = \tfrac{1}{4}b^3c_1n_1 - \tfrac{1}{12}b^3c_3n_3.$$

In the instrument under discussion the maker has taken $b = 2$ decimetres, $c_1 = \frac{1}{2}$ dec., $c_2 = c_3 = \frac{3}{5}$ dec.

$$\therefore A = n_1, \quad M = \tfrac{3}{5}n_2, \quad I = n_1 - \tfrac{2}{5}n_3.$$

The height $\bar{y}$ of the centre of gravity of the area above $OX = M/A$, and the moment of inertia with respect to an axis through the centre of gravity and parallel to $OX = I - A\bar{y}^2 = I - M^2/A$.

Results may be changed from decimetres to inches by multiplying $\bar{y}$ by a, A by a^2, M by a^3, and I by a^4, where $a = 3{\cdot}937$, the number of inches in a decimetre; $a^2 = 15{\cdot}500$, $a^3 = 61{\cdot}023$, $a^4 = 240{\cdot}290$.

MISCELLANEOUS EXAMPLES.

1. Prove Leibnitz's theorem for the nth differential of a product:

$$d^n(uv) = (d^n u)v + nd^{n-1}u\ dv + \frac{n(n-1)}{2!}d^{n-2}u\ d^2v + \ldots + u\,d^n v.$$

[By induction from $d(uv) = v\,du + u\,dv$.]

2. If $y^2 = a^2 + 2xy$, $d^2y/dx^2 = a^2/(y-x)^3$ and $d^2x/dy^2 = -a^2/y^3$.

3. The maximum value of $\left(\frac{1}{x}\right)^{x^2} = 1{\cdot}202$.

4. Show that the turning points of the curve $y = \sin x^{-1}$ are where $x = 2/(n\pi)$, n being any odd integer. (The number when $x = 0$ from any assigned value is therefore infinite.)

5. Given the volume of a right circular cylinder, show that the surface is a minimum when the altitude = the diameter.

6. The height of the greatest rectangle which can be inscribed in a given right segment of a parabola is two-thirds of the height of the segment.

7. Find the area of the rectangle circumscribing the loops of the curve $ay^3 - 3ax^2y = x^4$ (Fig. 36). *Ans.* $9a^2$.

8. $\int_0^{\frac{\pi}{4}} \sin^4\theta\ d\theta = {\cdot}045.$

9. $\int_0^{\frac{\pi}{4}} \tan^4\theta\ d\theta = {\cdot}119.$

10. $\int_0^{\frac{\pi}{4}} \sec^4\theta\ d\theta = 1\tfrac{1}{3}.$

11. $\int_1^{\infty} \frac{dx}{x^2 + x^4} = {\cdot}215.$

12. $\int_0^1 \frac{dx}{1 + x + x^2} = {\cdot}605.$

13. $\int_0^{\infty} \frac{dx}{(1+x^2)^3} = \frac{3\pi}{16}.$

14. $\int_0^a \frac{x^2\,dx}{(x^2+a^2)^{\frac{3}{2}}} = {\cdot}174.$

15. $\int_0^{\frac{\pi}{3}} \frac{dx}{\cos x} = 1{\cdot}317.$

16. $\int_0^{\frac{\pi}{3}} \frac{dx}{\cos^2 x} = 1{\cdot}732.$

17. $\int_0^{\frac{\pi}{3}} \frac{dx}{\cos^3 x} = 2{\cdot}391.$

18. $\int_0^{\frac{\pi}{3}} \frac{dx}{\cos^2 x} = 3{\cdot}464.$

19. $\int_{\cdot 1}^{\cdot 2} \sin^2 x\, dx = {\cdot}002.$

20. $\int_0^{\frac{1}{2}} \sec x\, dx = {\cdot}522.$

21. $\int_0^1 e^x \cos x\, dx = 1{\cdot}378.$

22. $\int_0^{\frac{\pi}{2}} \frac{dx}{(1+\cos x)^2} = \frac{2}{3}.$

23. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{\sin 2x} = {\cdot}275.$

24. $\int_0^{\frac{\pi}{2}} \frac{1-\cos^3 x}{\sin^2 x} dx = 2.$

25. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d\theta}{\sin^2\theta \cos\theta} = {\cdot}695.$

26. $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} = \pi.$

27. $\int_0^{\pi} \frac{dx}{2+\cos x} = 1{\cdot}813.$

28. $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = {\cdot}604.$

29. $\int_0^{\frac{\pi}{3}} \frac{dx}{\sin x + \cos x} = {\cdot}810.$

30. $\int_0^1 \frac{x\, dx}{(x+1)(x^2+1)} = {\cdot}219.$

31. $\int_0^1 x^2(1-x)^{\frac{1}{2}} dx = {\cdot}152.$

32. $\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}} dx = \frac{3\pi}{128}.$

33. $\int_0^a \sqrt{\frac{a-x}{x}} dx = \tfrac{1}{2}\pi a.$

34. $\int \frac{\cos \frac{1}{2}\theta}{\sin \theta} d\theta = \lambda\left(\frac{\theta-\pi}{2}\right).$

35. $\int \frac{d\theta}{\sec\theta + \tan\theta} = \lambda(\theta) + \log\cos\theta.$

36. $\int_0^{\infty} e^{-ax} \cos mx\, dx = \frac{a}{a^2+m^2}.$

37. $\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1} = \frac{\alpha}{2\sin\alpha}.$

38. $\int_0^{\pi} \left(\frac{x^2}{2\pi} - x\right) \cos mx\, dx = \frac{1}{m^2}.$

39. $\int \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan\theta\right).$

40. $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} = \frac{2}{3}\frac{(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}}{a-b}.$

41. $\mathcal{L}_{n=\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{2n^2}\right) = \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}.$

Let $1/n = dx$.

42. The area of the evolute of the ellipse $= \frac{3}{8}\frac{(a^2-b^2)^2}{ab}\pi.$

43. A cycloid revolves about its axis of symmetry. Show that the volume of the solid is $\pi a^3(\frac{3}{2}\pi^2-\frac{8}{3})$ and that the convex surface is $8\pi a^2(\pi-\frac{4}{3})$.

44. A hemispherical bowl of 1 ft. radius is filled with water which then runs out of an orifice at the bottom $\frac{1}{2}$ sq. in. in sectional area. Find the time of emptying, assuming that the velocity at the orifice $=\sqrt{2gx}$, where x is the height of the surface of the water above the orifice. *Ans.* 1 m. 46 s.

45. Find the centre of gravity of the area between the curve $(x/a)^{\frac{1}{2}}+(y/b)^{\frac{1}{2}}=1$ and the axes. *Ans.* $\overline{x}=\frac{1}{5}a$, $\overline{y}=\frac{1}{5}b$.

46. Prove that $\lambda(x)=x+\dfrac{x^3}{3!}+\dfrac{x^5}{4!}+\ldots$

47. $x\sin x=1-\frac{1}{2}\cos x-2\sum_{n=2}^{n=\infty}\dfrac{\cos n\pi\cos nx}{n^2-1}$, for $[-\pi,\pi]$.

48. $x\cos x=-\frac{1}{2}\sin x+2\sum_{n=2}^{n=\infty}\dfrac{(-1)^n n\sin nx}{n^2-1}$, for $]-\pi,\pi[$.

49. $x^3=\dfrac{\pi^3}{4}+\dfrac{6}{\pi}\sum_{n=1}^{n=\infty}\left[\dfrac{\pi^2}{n^2}\cos n\pi+\dfrac{2}{n^4}(1-\cos n\pi)\right]\cos nx$, for $[0,\pi]$.

Hence prove that $\dfrac{1}{1^4}+\dfrac{1}{3^4}+\dfrac{1}{5^4}+\ldots=\dfrac{\pi^4}{96}$, $\dfrac{1}{1^4}+\dfrac{1}{2^4}+\dfrac{1}{3^4}+\ldots=\dfrac{\pi^4}{90}$.

50. Find (1) the area, and (2) the length, of one loop of the curve $r^n=a^n\cos n\theta$.

Ans. (1) $\dfrac{\sqrt{\pi}\,a^2}{2}\dfrac{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}\right)}$, (2) $\dfrac{\sqrt{\pi}\,a}{n}\dfrac{\Gamma\left(\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2n}+\frac{1}{2}\right)}$.

51. Find the area in the first quadrant between the axes and the curve $\left(\dfrac{x}{a}\right)^n+\left(\dfrac{y}{b}\right)^n=1$. *Ans.* $\dfrac{\left[\Gamma\left(1+\frac{1}{n}\right)\right]^2}{\Gamma\left(1+\frac{2}{n}\right)}ab$.

Show that the area $\doteq ab$ when n is infinite.

52. Find the area of $x^3+y^3=a^3$ in the first quadrant, and the whole area of $x^4+y^4=a^4$. *Ans.* $0{\cdot}883a^2$, $3{\cdot}708a^2$.

53. Find the area of one oval of the curve $y^2=a^2\sin(x/b)$. *Ans.* $4{\cdot}792ab$.

54. Show that $\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{\sin\theta}\,d\theta\,.\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\sin\theta}}=\pi$.

55. The curve $r^m \cos m\theta = a^m$ rolls on a straight line. To find the differential equation of the locus of the polar origin.

Let the straight line be taken as x-axis, and the polar origin be at (x, y). Then r = the normal at (x, y), and y = the perpendicular on the tangent at (θ, r). Hence

$$u = \frac{1}{y\sqrt{1+p^2}}, \quad \frac{1}{y^2} = u^2 + \left(\frac{du}{d\theta}\right)^2, \quad a^m u^m = \cos m\theta.$$

Eliminate u and θ and show that $(1+p^2)^{1-m} = (y/a)^{2m}$.

56. A parabola rolls on a straight line. Show that the focus describes a catenary.

57. Find the centre of gravity of the arc of the quadrant of an ellipse (semi-axes a and b, eccentricity e).

$$Ans. \quad \bar{x} = \frac{a}{2}\left(1 + \frac{1}{2e}(1-e^2)\log\frac{1+e}{1-e}\right) / E(e, \tfrac{1}{2}\pi),$$

$$\bar{y} = \frac{b}{2}\left(\sqrt{1-e^2} + \frac{1}{e}\sin^{-1}e\right) / E(e, \tfrac{1}{2}\pi).$$

58. An ellipse (eccentricity e) and a circle have equal areas. Find the ratio of their circumferences. *Ans.* $\dfrac{2E(e, \frac{1}{2}\pi)}{\pi(1-e^2)^{\frac{1}{2}}}$.

59. Find the curves which make an angle α with the curves

$$r^n = a^n \cos n\theta, \quad r^n \cos n\theta = a^n.$$

$$Ans. \quad r^n = c^n \cos(n\theta + \alpha), \quad r^n \cos(n\theta - \alpha) = c^n.$$

60. Given $\dfrac{dx}{dt} - \dfrac{dy}{dt} + y = \sin 2t$, $\dfrac{dx}{dt} + \dfrac{dy}{dt} + x = 0$, show that

$$x = ce^{\frac{t}{\sqrt{2}}} + c_1 e^{-\frac{t}{\sqrt{2}}} - \tfrac{2}{5}\cos 2t,$$

$$y = -(\sqrt{2}+1)ce^{\frac{t}{\sqrt{2}}} + (\sqrt{2}-1)c_1 e^{-\frac{t}{\sqrt{2}}} + \tfrac{1}{5}\sin 2t + \tfrac{2}{5}\cos 2t.$$

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1. POWERS, NAPIERIAN LOGARITHMS, ETC.

x	x^{-1}	x^2	x^3	$x^{\frac{1}{2}}$	$(10x)^{\frac{1}{2}}$	$x^{\frac{3}{2}}$	$\log_e x$	$\log_e(10x)$
0·1	10	0·01	0·001	0·316	1·000	0·464	−2·303	0·000
0·2	5·000	0·04	0·008	0·447	1·414	0·585	−1·609	0·693
0·3	3·333	0·09	0·027	0·548	1·732	0·696	−1·204	1·099
0·4	2·500	0·16	0·064	0·632	2·000	0·737	−0·916	1·386
0·5	2·000	0·25	0·125	0·707	2·236	0·794	−0·693	1·609
0·6	1·667	0·36	0·216	0·775	2·449	0·843	−0·511	1·792
0·7	1·429	0·49	0·343	0·837	2·646	0·888	−0·357	1·946
0·8	1·250	0·64	0·512	0·894	2·828	0·928	−0·223	2·079
0·9	1·111	0·81	0·729	0·949	3·000	0·965	−0·105	2·197
1·0	1·000	1·00	1·000	1·000	3·162	1·000	0·000	2·303
1·1	0·909	1·21	1·331	1·049	3·317	1·032	0·095	2·398
1·2	0·833	1·44	1·728	1·095	3·464	1·063	0·182	2·485
1·3	0·769	1·69	2·197	1·140	3·606	1·091	0·262	2·565
1·4	0·714	1·96	2·744	1·183	3·742	1·119	0·336	2·639
1·5	0·667	2·25	3·375	1·225	3·873	1·145	0·405	2·708
1·6	0·625	2·56	4·096	1·265	4·000	1·170	0·470	2·773
1·7	0·588	2·89	4·913	1·304	4·123	1·193	0·531	2·833
1·8	0·556	3·24	5·832	1·342	4·243	1·216	0·588	2·890
1·9	0·526	3·61	6·859	1·378	4·359	1·239	0·642	2·944
2·0	0·500	4·00	8·000	1·414	4·472	1·260	0·693	2·996
2·1	0·476	4·41	9·261	1·449	4·583	1·281	0·742	3·045
2·2	0·455	4·84	10·648	1·483	4·690	1·301	0·788	3·091
2·3	0·435	5·29	12·167	1·517	4·796	1·320	0·833	3·135
2·4	0·417	5·76	13·824	1·549	4·899	1·339	0·875	3·178
2·5	0·400	6·25	15·625	1·581	5·000	1·357	0·916	3·219
2·6	0·385	6·76	17·576	1·612	5·099	1·375	0·956	3·258
2·7	0·370	7·29	19·683	1·643	5·196	1·392	0·993	3·296
2·8	0·357	7·84	21·952	1·673	5·292	1·410	1·030	3·332
2·9	0·345	8·41	24·389	1·703	5·385	1·426	1·065	3·367
3·0	0·333	9·00	27·000	1·732	5·477	1·442	1·099	3·401
3·1	0·323	9·61	29·791	1·761	5·568	1·458	1·131	3·434
3·2	0·313	10·24	32·768	1·789	5·657	1·474	1·163	3·466
3·3	0·303	10·89	35·937	1·817	5·745	1·489	1·194	3·497
3·4	0·294	11·56	39·304	1·844	5·831	1·504	1·224	3·526
3·5	0·286	12·25	42·875	1·871	5·916	1·518	1·253	3·555
3·6	0·278	12·96	46·656	1·897	6·000	1·533	1·281	3·584
3·7	0·270	13·69	50·653	1·924	6·083	1·547	1·308	3·611
3·8	0·263	14·44	54·872	1·949	6·164	1·561	1·335	3·638
3·9	0·256	15·21	59·319	1·975	6·245	1·574	1·361	3·664
4·0	0·250	16·00	64·000	2·000	6·325	1·587	1·386	3·689
4·1	0·244	16·81	68·921	2·025	6·403	1·601	1·411	3·714
4·2	0·238	17·64	74·088	2·049	6·481	1·613	1·435	3·738
4·3	0·233	18·49	79·507	2·074	6·557	1·626	1·459	3·761
4·4	0·227	19·36	85·184	2·098	6·633	1·639	1·482	3·784
4·5	0·222	20·25	91·125	2·121	6·708	1·651	1·504	3·807
4·6	0·217	21·16	97·336	2·145	6·782	1·663	1·526	3·829
4·7	0·213	22·09	103·823	2·168	6·856	1·675	1·548	3·850
4·8	0·208	23·04	110·592	2·191	6·928	1·687	1·569	3·871
4·9	0·204	24·01	117·649	2·214	7·000	1·699	1·589	3·892
5·0	0·200	25·00	125·000	2·236	7·071	1·710	1·609	3·912

$e = 2{\cdot}71828$. $\log_e 10 = 2{\cdot}30259$, $\log_{10} e = 0{\cdot}43429$.

1. POWERS, NAPIERIAN LOGARITHMS, ETC.

x	x^{-1}	x^2	x^3	$x^{\frac{1}{2}}$	$(10x)^{\frac{1}{2}}$	$x^{\frac{1}{3}}$	$\log_e x$	$\log_e(10x)$
5·0	0·200	25·00	125·000	2·236	7·071	1·710	1·609	3·912
5·1	0·196	26·01	132·651	2·258	7·141	1·721	1·629	3·932
5·2	0·192	27·04	140·608	2·280	7·211	1·732	1·649	3·951
5·3	0·189	28·09	148·877	2·302	7·280	1·744	1·668	3·970
5·4	0·185	29·16	157·464	2·324	7·348	1·754	1·686	3·989
5·5	0·182	30·25	166·375	2·345	7·416	1·765	1·705	4·007
5·6	0·179	31·36	175·616	2·366	7·483	1·776	1·723	4·025
5·7	0·175	32·49	185·193	2·387	7·550	1·786	1·740	4·043
5·8	0·172	33·64	195·112	2·408	7·616	1·797	1·758	4·060
5·9	0·169	34·81	205·379	2·429	7·681	1·807	1·775	4·078
6·0	0·167	36·00	216·000	2·449	7·746	1·817	1·792	4·094
6·1	0·164	37·21	226·981	2·470	7·810	1·827	1·808	4·111
6·2	0·161	38·44	238·328	2·490	7·874	1·837	1·825	4·127
6·3	0·159	39·69	250·047	2·510	7·937	1·847	1·841	4·143
6·4	0·156	40·96	262·144	2·530	8·000	1·857	1·856	4·159
6·5	0·154	42·25	274·625	2·550	8·062	1·866	1·872	4·174
6·6	0·152	43·56	287·496	2·569	8·124	1·876	1·887	4·190
6·7	0·149	44·89	300·763	2·588	8·185	1·885	1·902	4·205
6·8	0·147	46·24	314·432	2·608	8·246	1·895	1·917	4·220
6·9	0·145	47·61	328·509	2·627	8·307	1·904	1·932	4·234
7·0	0·143	49·00	343·000	2·646	8·367	1·913	1·946	4·248
7·1	0·141	50·41	357·911	2·665	8·426	1·922	1·960	4·263
7·2	0·139	51·84	373·248	2·683	8·485	1·931	1·974	4·277
7·3	0·137	53·29	389·017	2·702	8·544	1·940	1·988	4·290
7·4	0·135	54·76	405·224	2·720	8·602	1·949	2·001	4·304
7·5	0·133	56·25	421·875	2·739	8·660	1·957	2·015	4·317
7·6	0·132	57·76	438·976	2·757	8·718	1·966	2·028	4·331
7·7	0·130	59·29	456·533	2·775	8·775	1·975	2·041	4·344
7·8	0·128	60·84	474·552	2·793	8·832	1·983	2·054	4·357
7·9	0·127	62·41	493·039	2·811	8·888	1·992	2·067	4·369
8·0	0·125	64·00	512·000	2·828	8·944	2·000	2·079	4·382
8·1	0·123	65·61	531·441	2·846	9·000	2·008	2·092	4·394
8·2	0·122	67·24	551·368	2·864	9·055	2·017	2·104	4·407
8·3	0·120	68·89	571·787	2·881	9·110	2·025	2·116	4·419
8·4	0·119	70·56	592·704	2·898	9·165	2·032	2·128	4·431
8·5	0·118	72·25	614·125	2·915	9·220	2·041	2·140	4·443
8·6	0·116	73·96	636·056	2·933	9·274	2·040	2·152	4·454
8·7	0·115	75·69	658·503	2·950	9·327	2·057	2·163	4·466
8·8	0·114	77·44	681·472	2·966	9·381	2·065	2·175	4·477
8·9	0·112	79·21	704·969	2·983	9·434	2·072	2·186	4·489
9·0	0·111	81·00	729·000	3·000	9·487	2·080	2·197	4·500
9·1	0·110	82·81	753·571	3·017	9·539	2·088	2·208	4·511
9·2	0·109	84·64	778·688	3·033	9·592	2·095	2·219	4·522
9·3	0·108	86·49	804·357	3·050	9·644	2·103	2·230	4·533
9·4	0·106	88·36	830·584	3·066	9·695	2·110	2·241	4·543
9·5	0·105	90·25	857·375	3·082	9·747	2·118	2·251	4·554
9·6	0·104	92·16	884·736	3·098	9·798	2·125	2·262	4·564
9·7	0·103	94·09	912·673	3·114	9·849	2·133	2·272	4·575
9·8	0·102	96·04	941·192	3·130	9·899	2·140	2·282	4·585
9·9	0·101	98·01	970·299	3·146	9·950	2·147	2·293	4·595

$\log_{10} x = 0{\cdot}43429 \log_e x$, $\log_e x = 2{\cdot}30259 \log_{10} x$.

2. CIRCULAR FUNCTIONS I.

θ Degrees	θ Radians	$\sin\theta$	$\operatorname{cosec}\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\cos\theta$		
0	0·000	0·000	∞	0·000	∞	1·000	1·000	1·571	90
1	0·017	0·017	57·299	0·017	57·290	1·000	1·000	1·553	89
2	0·035	0·035	28·654	0·035	28·636	1·001	0·999	1·536	88
3	0·052	0·052	19·107	0·052	19·081	1·001	0·999	1·518	87
4	0·070	0·070	14·336	0·070	14·301	1·002	0·998	1·501	86
5	0·087	0·087	11·474	0·087	11·430	1·004	0·996	1·484	85
6	0·105	0·105	9·567	0·105	9·514	1·006	0·995	1·466	84
7	0·122	0·122	8·206	0·123	8·144	1·008	0·993	1·449	83
8	0·140	0·139	7·185	0·141	7·115	1·010	0·990	1·431	82
9	0·157	0·156	6·392	0·158	6·314	1·012	0·988	1·414	81
10	0·175	0·174	5·759	0·176	5·671	1·015	0·985	1·396	80
11	0·192	0·191	5·241	0·194	5·145	1·019	0·982	1·379	79
12	0·209	0·208	4·810	0·213	4·705	1·022	0·978	1·361	78
13	0·227	0·225	4·445	0·231	4·331	1·026	0·974	1·344	77
14	0·244	0·242	4·134	0·249	4·011	1·031	0·970	1·326	76
15	0·262	0·259	3·864	0·268	3·732	1·035	0·966	1·309	75
16	0·279	0·276	3·628	0·287	3·487	1·040	0·961	1·292	74
17	0·297	0·292	3·420	0·306	3·271	1·046	0·956	1·274	73
18	0·314	0·309	3·236	0·325	3·078	1·051	0·951	1·257	72
19	0·332	0·326	3·072	0·344	2·904	1·058	0·946	1·239	71
20	0·349	0·342	2·924	0·364	2·747	1·064	0·940	1·222	70
21	0·367	0·358	2·790	0·384	2·605	1·071	0·934	1·204	69
22	0·384	0·375	2·669	0·404	2·475	1·079	0·927	1·187	68
23	0·401	0·391	2·559	0·424	2·356	1·086	0·921	1·169	67
24	0·419	0·407	2·459	0·445	2·246	1·095	0·914	1·152	66
25	0·436	0·423	2·366	0·466	2·145	1·103	0·906	1·134	65
26	0·454	0·438	2·281	0·488	2·050	1·113	0·899	1·117	64
27	0·471	0·454	2·203	0·510	1·963	1·122	0·891	1·100	63
28	0·489	0·469	2·130	0·532	1·881	1·133	0·883	1·082	62
29	0·506	0·485	2·063	0·554	1·804	1·143	0·875	1·065	61
30	0·524	0·500	2·000	0·577	1·732	1·155	0·866	1·047	60
31	0·541	0·515	1·942	0·601	1·664	1·167	0·857	1·030	59
32	0·559	0·530	1·887	0·625	1·600	1·179	0·848	1·012	58
33	0·576	0·545	1·836	0·649	1·540	1·192	0·839	0·995	57
34	0·593	0·559	1·788	0·675	1·483	1·206	0·829	0·977	56
35	0·611	0·574	1·743	0·700	1·428	1·221	0·819	0·960	55
36	0·628	0·588	1·701	0·727	1·376	1·236	0·809	0·942	54
37	0·646	0·602	1·662	0·754	1·327	1·252	0·799	0·925	53
38	0·663	0·616	1·624	0·781	1·280	1·269	0·788	0·908	52
39	0·681	0·629	1·589	0·810	1·235	1·287	0·777	0·890	51
40	0·698	0·643	1·556	0·839	1·192	1·305	0·766	0·873	50
41	0·716	0·656	1·524	0·869	1·150	1·325	0·755	0·855	49
42	0·733	0·669	1·494	0·900	1·111	1·346	0·743	0·838	48
43	0·750	0·682	1·466	0·933	1·072	1·367	0·731	0·820	47
44	0·768	0·695	1·440	0·966	1·036	1·390	0·719	0·803	46
45	0·785	0·707	1·414	1·000	1·000	1·414	0·707	0·785	45
		$\cos\theta$	$\sec\theta$	$\cot\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sin\theta$	θ Radians	θ Degrees

1 rdn. = 57°·29578, 1° = 0·01745 rdn., 1 rdn. = 206265″, 1″ = ·000004848 rdn.

3. CIRCULAR FUNCTIONS, II.

θ Radians	θ Degrees	$\sin\theta$	$\operatorname{cosec}\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\cos\theta$
0·00	0·00	0·000	∞	0·000	∞	1·000	1·000
0·01	0·57	0·010	100·002	0·010	99·997	1·000	1·000
0·02	1·15	0·020	50·003	0·020	49·993	1·000	1·000
0·03	1·72	0·030	33·338	0·030	33·323	1·000	1·000
0·04	2·29	0·040	25·007	0·040	24·987	1·001	0·999
0·05	2·86	0·050	20·008	0·050	19·983	1·001	0·999
0·06	3·44	0·060	16·677	0·060	16·647	1·002	0·998
0·07	4·01	0·070	14·297	0·070	14·263	1·002	0·998
0·08	4·58	0·080	12·513	0·080	12·473	1·003	0·997
0·09	5·16	0·090	11·126	0·090	11·081	1·004	0·996
0·1	5·73	0·100	10·017	0·100	9·967	1·005	0·995
0·2	11·46	0·199	5·033	0·203	4·933	1·020	0·980
0·3	17·19	0·296	3·384	0·309	3·232	1·047	0·955
0·4	22·92	0·389	2·568	0·423	2·365	1·086	0·921
0·5	28·65	0·479	2·086	0·546	1·830	1·140	0·878
0·6	34·38	0·565	1·771	0·684	1·462	1·212	0·825
0·7	40·11	0·644	1·552	0·842	1·187	1·307	0·765
0·8	45·84	0·717	1·394	1·030	0·971	1·435	0·697
0·9	51·57	0·783	1·277	1·260	0·794	1·609	0·622
1·0	57·30	0·842	1·188	1·558	0·642	1·851	0·540
1·1	63·03	0·891	1·122	1·965	0·509	2·205	0·453
1·2	68·75	0·932	1·073	2·571	0·389	2·760	0·362
1·3	74·48	0·964	1·038	3·602	0·278	3·738	0·268
1·4	80·21	0·985	1·015	5·798	0·172	5·884	0·170
1·5	85·94	0·997	1·003	14·101	0·071	14·136	0·071
$\frac{1}{2}\pi$	90·00	1·000	1·000	∞	0·000	∞	0·000

$\pi = 3{\cdot}14159$, $\pi^2 = 9{\cdot}86960$, $\sqrt{\pi} = 1{\cdot}77245$,
$\pi^{-1} = 0{\cdot}31831$, $\pi^{-2} = 0{\cdot}10132$, $\sqrt{\pi^{-1}} = 0{\cdot}56419$.

4. HYPERBOLIC FUNCTIONS.

x	e^x	e^{-x}	sinh x	cosh x	tanh x	coth x	sech x	cosech x
0·0	1·000	1·000	0·000	1·000	0·000	∞	1·000	∞
0·1	1·105	0·905	0·100	1·005	0·100	10·035	0·995	9·985
0·2	1·221	0·819	0·201	1·020	0·197	5·067	0·980	4·975
0·3	1·350	0·741	0·305	1·045	0·291	3·433	0·957	3·284
0·4	1·492	0·670	0·411	1·081	0·380	2·632	0·925	2·433
0·5	1·649	0·607	0·521	1·128	0·462	2·164	0·887	1·919
0·6	1·822	0·549	0·637	1·185	0·537	1·862	0·844	1·570
0·7	2·014	0·497	0·759	1·255	0·604	1·655	0·796	1·318
0·8	2·226	0·449	0·888	1·337	0·664	1·506	0·748	1·126
0·9	2·460	0·407	1·027	1·433	0·716	1·396	0·698	0·974
1·0	2·718	0·368	1·175	1·543	0·762	1·313	0·648	0·851
1·1	3·004	0·333	1·336	1·669	0·800	1·249	0·599	0·749
1·2	3·320	0·301	1·509	1·811	0·834	1·200	0·552	0·662
1·3	3·669	0·273	1·698	1·971	0·862	1·161	0·507	·589
1·4	4·055	0·247	1·904	2·151	0·885	1·129	0·465	0·525
1·5	4·482	0·223	2·129	2·352	0·905	1·105	0·425	0·470
1·6	4·953	0·202	2·376	2·577	0·922	1·085	0·388	0·421
1·7	5·474	0·183	2·646	2·828	0·935	1·069	0·354	0·378
1·8	6·050	0·165	2·942	3·107	0·947	1·056	0·322	0·340
1·9	6·686	0·150	3·268	3·418	0·956	1·046	0·293	0·306
2·0	7·389	0·135	3·627	3·762	0·964	1·037	0·266	0·276
2·1	8·166	0·122	4·022	4·144	0·970	1·031	0·241	0·249
2·2	9·025	0·111	4·457	4·568	0·976	1·025	0·219	0·224
2·3	9·974	0·100	4·937	5·037	0·980	1·020	0·198	0·203
2·4	11·023	0·091	5·466	5·557	0·984	1·017	0·180	0·183
2·5	12·182	0·082	6·050	6·132	0·987	1·014	0·163	0·165
2·6	13·464	0·074	6·695	6·769	0·989	1·011	0·148	0·149
2·7	14·880	0·067	7·406	7·473	0·991	1·009	0·134	0·135
2·8	16·445	0·061	8·192	8·253	0·993	1·007	0·121	0·122
2·9	18·174	0·055	9·060	9·115	0·994	1·006	0·110	0·110
3·0	20·090	0·050	10·018	10·068	0·995	1·005	0·099	0·099
3·1	22·198	0·045	11·076	11·122	0·996	1·004	0·090	0·090
3·2	24·533	0·041	12·246	12·287	0·997	1·003	0·081	0·082
3·3	27·113	0·037	13·538	13·575	0·997	1·003	0·074	0·074
3·4	29·964	0·033	14·965	14·999	0·998	1·002	0·067	0·067
3·5	33·116	0·030	16·543	16·573	0·998	1·002	0·060	0·060
3·6	36·598	0·027	18·286	18·313	0·999	1·001	0·055	0·055
3·7	40·447	0·025	20·211	20·236	0·999	1·001	0·049	0·049
3·8	44·701	0·022	22·339	22·362	0·999	1·001	0·045	0·045
3·9	49·402	0·020	24·691	24·711	0·999	1·001	0·040	0·041
4·0	54·598	0·018	27·290	27·308	0·999	1·001	0·037	0·037
$\frac{1}{2}\pi$	4·810	0·208	2·301	2·509	0·917	1·091	0·398	0·435
π	23·141	0·043	11·549	11·592	0·996	1·004	0·087	0·087

If $x>4$, then (approximately) $\sinh x=\cosh x=\frac{1}{2}e^x=\frac{1}{2}$ the Napierian antilogarithm of x.

5. LAMBDA FUNCTION.

θ	$\lambda(\theta)$	θ	$\lambda(\theta)$	θ	$\lambda(\theta)$	θ	$\lambda(\theta)$	θ	$\lambda(\theta)$	θ	$\lambda(\theta)$
0°	0·000	15°	0·265	30°	0·549	45°	0·881	60°	1·317	75°	2·028
1°	0·017	16°	0·283	31°	0·570	46°	0·906	61°	1·352	76°	2·097
2°	0·035	17°	0·301	32°	0·590	47°	0·932	62°	1·389	77°	2·172
3°	0·052	18°	0·319	33°	0·611	48°	0·957	63°	1·427	78°	2·253
4°	0·070	19°	0·338	34°	0·632	49°	0·984	64°	1·466	79°	2·340
5°	0·087	20°	0·356	35°	0·653	50°	1·011	65°	1·506	80°	2·436
6°	0·105	21°	0·375	36°	0·674	51°	1·038	66°	1·549	81°	2·542
7°	0·122	22°	0·394	37°	0·696	52°	1·066	67°	1·592	82°	2·660
8°	0·140	23°	0·413	38°	0·718	53°	1·095	68°	1·638	83°	2·794
9°	0·158	24°	0·432	39°	0·740	54°	1·124	69°	1·686	84°	2·949
10°	0·175	25°	0·451	40°	0·763	55°	1·154	70°	1·735	85°	3·131
11°	0·193	26°	0·470	41°	0·786	56°	1·185	71°	1·788	86°	3·355
12°	0·211	27°	0·490	42°	0·809	57°	1·217	72°	1·843	87°	3·643
13°	0·229	28°	0·509	43°	0·833	58°	1·249	73°	1·901	88°	4·048
14°	0·247	29°	0·529	44°	0·857	59°	1·283	74°	1·962	89°	4·741
15°	0·265	30°	0·549	45°	0·881	60°	1·317	75°	2·028	90°	∞

$\lambda(\theta) = \log_e \tan(\frac{1}{4}\pi + \frac{1}{2}\theta) = \log_e (\sec\theta + \tan\theta)$, $\lambda(-\theta) = -\lambda(\theta)$, $\theta = \mathrm{gd}\,\lambda(\theta)$.

6. GAMMA FUNCTION.

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
1·00	1	1·20	0·9182	1·40	0·8873	1·60	0·8935	1·80	0·9314
1·01	0·9943	1·21	0·9156	1·41	0·8868	1·61	0·8947	1·81	0·9341
1·02	0·9888	1·22	0·9131	1·42	0·8864	1·62	0·8959	1·82	0·9369
1·03	0·9836	1·23	0·9108	1·43	0·8860	1·63	0·8972	1·83	0·9397
1·04	0·9784	1·24	0·9085	1·44	0·8858	1·64	0·8986	1·84	0·9426
1·05	0·9735	1·25	0·9064	1·45	0·8856	1·65	0·9001	1·85	0·9456
1·06	0·9688	1·26	0·9044	1·46	0·8856	1·66	0·9017	1·86	0·9487
1·07	0·9642	1·27	0·9025	1·47	0·8856	1·67	0·9033	1·87	0·9518
1·08	0·9597	1·28	0·9007	1·48	0·8857	1·68	0·9050	1·88	0·9551
1·09	0·9555	1·29	0·8990	1·49	0·8860	1·69	0·9068	1·89	0·9584
1·10	0·9514	1·30	0·8975	1·50	0·8862	1·70	0·9086	1·90	0·9618
1·11	0·9474	1·31	0·8960	1·51	0·8866	1·71	0·9106	1·91	0·9652
1·12	0·9436	1·32	0·8946	1·52	0·8870	1·72	0·9126	1·92	0·9688
1·13	0·9399	1·33	0·8934	1·53	0·8876	1·73	0·9147	1·93	0·9724
1·14	0·9364	1·34	0·8922	1·54	0·8882	1·74	0·9168	1·94	0·9761
1·15	0·9330	1·35	0·8911	1·55	0·8889	1·75	0·9191	1·95	0·9799
1·16	0·9298	1·36	0·8902	1·56	0·8896	1·76	0·9214	1·96	0·9837
1·17	0·9267	1·37	0·8893	1·57	0·8905	1·77	0·9238	1·97	0·9877
1·18	0·9237	1·38	0·8885	1·58	0·8914	1·78	0·9262	1·98	0·9917
1·19	0·9209	1·39	0·8879	1·59	0·8924	1·79	0·9288	1·99	0·9958
1·20	0·9182	1·40	0·8873	1·60	0·8935	1·80	0·9314	2·00	1

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx, \qquad \Gamma(n+1) = n\Gamma(n).$$

7. FIRST ELLIPTIC INTEGRAL, $F(m, \theta)$, $m = \sin \alpha$.

θ	$\alpha = 0°$	$\alpha = 15°$	$\alpha = 30°$	$\alpha = 45°$	$\alpha = 60°$	$\alpha = 75°$	$\alpha = 90°$
0°	0·000	0·000	0·000	0·000	0·000	0·000	0·000
5°	0·087	0·087	0·087	0·087	0·087	0·087	0·087
10°	0·175	0·175	0·175	0·175	0·175	0·175	0·175
15°	0·262	0·262	0·263	0·263	0·264	0·265	0·265
20°	0·349	0·350	0·351	0·353	0·354	0·356	0·356
25°	0·436	0·437	0·440	0·443	0·447	0·450	0·451
30°	0·524	0·525	0·529	0·536	0·542	0·547	0·549
35°	0·611	0·613	0·620	0·630	0·641	0·649	0·653
40°	0·698	0·702	0·712	0·727	0·744	0·757	0·763
45°	0·785	0·790	0·804	0·826	0·851	0·873	0·881
50°	0·873	0·879	0·898	0·928	0·965	0·997	1·011
55°	0·960	0·968	0·993	1·034	1·085	1·133	1·154
60°	1·047	1·058	1·090	1·142	1·213	1·284	1·317
65°	1·134	1·147	1·187	1·254	1·349	1·453	1·506
70°	1·222	1·237	1·285	1·370	1·494	1·647	1·735
75°	1·309	1·327	1·385	1·488	1·649	1·871	2·028
80°	1·396	1·418	1·485	1·608	1·813	2·134	2·436
85°	1·484	1·508	1·585	1·731	1·983	2·437	3·131
90°	1·571	1·598	1·686	1·854	2·157	2·768	∞

8. SECOND ELLIPTIC INTEGRAL, $E(m, \theta)$, $m = \sin \alpha$.

θ	$\alpha = 0°$	$\alpha = 15°$	$\alpha = 30°$	$\alpha = 45°$	$\alpha = 60°$	$\alpha = 75°$	$\alpha = 90°$
0°	0·000	0·000	0·000	0·000	0·000	0·000	0·000
5°	0·087	0·087	0·087	0·087	0·087	0·087	0·087
10°	0·175	0·174	0·174	0·174	0·174	0·174	0·174
15°	0·262	0·262	0·261	0·260	0·260	0·259	0·259
20°	0·349	0·349	0·347	0·346	0·344	0·342	0·342
25°	0·436	0·435	0·433	0·430	0·426	0·424	0·423
30°	0·524	0·522	0·518	0·512	0·506	0·502	0·500
35°	0·611	0·608	0·602	0·593	0·583	0·576	0·574
40°	0·698	0·695	0·685	0·672	0·657	0·647	0·643
45°	0·785	0·781	0·767	0·748	0·728	0·713	0·707
50°	0·873	0·866	0·848	0·823	0·795	0·774	0·766
55°	0·960	0·952	0·928	0·895	0·859	0·830	0·819
60°	1·047	1·037	1·008	0·965	0·918	0·881	0·866
65°	1·134	1·122	1·086	1·033	0·974	0·926	0·906
70°	1·222	1·206	1·163	1·099	1·027	0·965	0·940
75°	1·309	1·291	1·240	1·163	1·076	0·999	0·966
80°	1·396	1·375	1·316	1·227	1·122	1·028	0·985
85°	1·484	1·460	1·392	1·289	1·167	1·053	0·996
90°	1·571	1·541	1·467	1·351	1·211	1·075	1·000

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